Lecture Notes: Quantitative Reasoning and Mathematical Thinking[1]

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Chapter 1

Introduction

Courant and Robbins, in What is Mathematics? (1941), present mathematics not as a dry collection of formulas and tools, but as a living, creative discipline rooted in human thought and curiosity. For them, mathematics is both a pathway for understanding the natural world and an autonomous intellectual pursuit that reveals structures of order, beauty, and generality. They stress that its essence lies in the interplay between abstraction and concrete problem-solving: starting from simple, practical problems, mathematics ascends to general concepts and theories that then illuminate new domains.

They emphasize accessibility and unity: mathematics belongs to everyone who is willing to think rigorously, and its spirit combines logic with imagination. Rather than reducing it to calculation or technical skill, Courant and Robbins describe mathematics as "an expression of the human mind" where precision, creativity, and aesthetic appreciation converge. Their central idea is that mathematics is at once useful, philosophical, and artistic — simultaneously a language of science, a training ground for reasoning, and a source of intellectual delight.

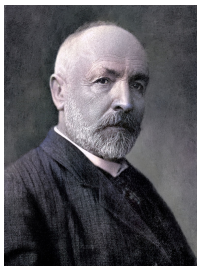
Early mathematics was computational when the emphasis was on finding methods to obtain solutions. However, over the years, the disciplines of mathematics and computer science – the subject of designing algorithms for problem solving – have diverged. In mathematics abstraction is symbolic and logical. It seeks general structures, patterns, and proofs independent of implementation. It often endeavours to seek and capture common structures across different abstractions. The primary aim is truth and understanding – developing rigorous proofs, ensuring logical consistency, and uncovering general laws. Utility often follows from this pursuit but is not always the main driver. In contrast, the role of abstraction in computational thinking is more operational and algorithmic. It emphasizes creating computational process models for natural, social and even abstract phenomena for operational analysis. The primary aim is to construct effective procedures – designing algorithms that solve problems efficiently, often under constraints of time, memory, and real-world complexity. The power lies in execution and exploration — trying to reveal insights about systems too complex to solve analytically. Both have become fundamental strands of epistemology that are essential for critical scientific thinking.

Data-driven inference represents a third way of knowing, distinct from the deductive rigour of mathematics and the constructive procedures of computational thinking. As practiced in modern data science and machine learning, it seeks knowledge not by proving theorems or designing explicit algorithms, but by discovering patterns and regularities directly from empirical data. Its epistemic core is induction at scale: hypotheses, models, or predictors are justified by their ability to capture hidden correlations and to generalize to new observations. Unlike mathematics, correctness is not absolute, and unlike computational thinking, procedures are not always fully transparent. Instead, credibility arises from empirical adequacy — the degree to which models explain, predict, or align with observed phenomena. This mode of inference expands our epistemic toolkit for a world where complexity and abundance of data overwhelm deductive or constructive methods, but it also brings new philosophical challenges: uncertainty about correctness, bias, and the gap between correlation and causation.

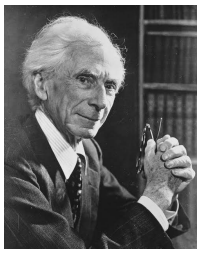
In this course we will try to cover some fundamentals of all of the above.

Chapter 2

God gave us numbers, and human thought created algorithms



"In mathematics the art of proposing a question must be held of higher value than solving it."
"A set is a Many that allows itself to be thought of as a One." – Georg Cantor



"A number will be a set of classes such as that any two are similar to each other, and none outside the set are similar to any inside the set."
"Mathematics rightly viewed possesses not only truth but supreme beauty." – Bertrand Russel

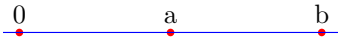
2.1 Numbers

Our discussion on mathematics and computing must start with numbers. What are numbers after all? The same number may be represented with symbols such as 3, III, or even as a line of a fixed length. But what is the underlying concept behind the different representations?

Bertrand Russell defined numbers as sizes (or *cardinality*) of *collections*. Some examples of *equinumerous* collections are $\{Red, Blue, Green\}$, $\{Amir, Salman, Shahrukh\}$, $\{Godavari, Kaveri, Krishna\}$. *Collections* are also called *Sets* or *Classes* in Mathematics. All three Sets above are of cardinality 3.

Russel defined a number as – *the number of a class is the class of all those classes that are similar (equinumerous) to it.* So, according to Russel, a number is a class of classes.

Note that *cardinality* of sets is not the only way to describe the concept of a number. A number may also be a measure of a length. For example, in the straight line below, if we define the segment $\overline{0a}$ to be the *unit length* representing the number 1, then the line segment $\overline{0b}$ which is twice the length of $\overline{0a}$ may represent the number 2.



Without belabouring the point, it will suffice to say for our purpose that all of us intuitively understand what numbers mean.

2.1.1 Numbers may be represented in multiple ways

However, we need to do useful stuff with numbers – we need to add, subtract, multiply and divide them for obvious practical reasons. Indeed, the history of numbers date back to the Mesolithic stone age. The early humans had to figure out – due to a variety of practical considerations – that if they put two similar collections of size two and size three together, the larger collection becomes of size five.

Civilisations have found many ways to represent numbers through the ages. Some examples are as tally marks in the prehistoric to early civilisations – as straight marks on bones, sticks, or stones – as can be observed in the archaeological evidence of the Ishango bones from around 20000 BCE; as Egyptian – a stroke for 1, heel bone for 10, coil of rope for 100, etc. – or Roman – I, V, X, L, C, D, M – numerals; as Base-60 (sexagesimal) numbers written as combinations of "1" and "10" wedges by the Babylonians around 2000 BCE; as used rods arranged on counting boards in base-10 with positional notation in Chinese rod systems; as positional decimal systems in Indian numerals in the Gupta period around 5^{th} century CE; as beads or stones moved on rods or grooves to represent numbers in Abacus systems in China, Rome, Mesopotamia and Jerusalem; with Indo-Arabic numerals in the medieval period; with various mechanical calculators such as Napier's bones, Slide rules, Pascal's calculator, and Leibniz's stepped reckoner in the 17^{th} century; as gears and levers in Charles Babbage's first programmable computer – the Analytic Engine; and as bits and bytes in modern digital computers. Note, also, that the methods of carrying out these operations – the algorithms – will necessarily depend on the representation we choose for numbers.

2.2 Sets

We will use *Sets* quite a bit in this course. We may describe a *Set* or a *Collection* by explicitly listing out its elements without duplicates, such as in the examples above. We may sometimes also describe a Set with a property like "all students enrolled in the QRMT section FC-0306-3". We write this formally using a variable x as $\{x \mid x \text{ is a student in the QRMT section FC-0306-3}\}$. The symbol $\mid$ is read as "such that".

If an element x belongs to a set A, we usually write this as $x \in A$.

Here are some more examples of Sets:

1. $A = \{x \mid x \text{ is a student pursuing a degree in India}\}$
2. $B = \{x \mid x \text{ is a CS Major student at Ashoka University}\}$
3. $C = \{x \mid x \text{ is a CS Major student at Ashoka University } \textbf{and } x \text{ is female}\}$

Clearly, all members of C are also members of B, and all members of B are members of A. We then say that C is a *subset* of B ($C \subseteq B$), and B is a *subset* of A ($B \subseteq A$). Formally, a set B is a *subset* of another set A, denoted as $B \subseteq A$, if $x \in A$ whenever $x \in B$. The empty set is denoted by ϕ, its size is zero (0), and it is a subset of all sets.

Given two sets A and B, the *union* $A \cup B$ is the set of all elements that are in A, or in B, or in both. Formally, $A \cup B = \{x \mid x \in A \text{ or } x \in B\}$. For example, if $A = \{1, 2, 3\}$ and $B = \{3, 4, 5\}$, then $A \cup B = \{1, 2, 3, 4, 5\}$.

Given two sets A and B, the *intersection* $A \cap B$ is the set of all elements that are in both A and B. Formally, $A \cap B = \{x \mid x \in A \text{ and } x \in B\}$. For example, if $A = \{1, 2, 3\}$ and $B = \{3, 4, 5\}$, then $A \cap B = \{3\}$.

Clearly, for any set A, $A \cup \phi = A$ and $A \cap \phi = \phi$,

Exercise 2.1 *Suppose $B \subseteq A$. Argue that*

1. $A \cup B = A$
2. $A \cap B = B$

2.3 The set of Natural numbers

Some sets can also be unbounded or infinite. We define the set of *Natural numbers* as $\mathbb{N} = \{0, 1, 2, 3, \ldots\}$[1].

While we all intuitively understand this set, note that the elements of the set are as yet uninterpreted and undefined. We can overcome this lacunae by assuming a *God-gifted* ability to count. Given a number n as the size of a Set or a length, let us assume that we can interpret and construct the successor of n as $S(n) = n + 1$. Then, we can formally define the set of Natural numbers $\mathbb{N}$ as

1. $0 \in \mathbb{N}$, where 0 is the symbol that denotes the size of the empty set, and
2. if $n \in \mathbb{N}$, then $S(n) = n + 1 \in \mathbb{N}$

We can then adopt a suitable representation for successive elements in the set $\mathbb{N}$. Note that the set $\mathbb{N}$ is unbounded, because every number – no matter how large – has a successor.

2.3.1 Addition

We observed that the underlying concept of a number is independent of specific representations. Ideally, so should be the concepts of carrying out various operations with numbers. We may think of addition – the sum $a + b$ of two numbers a and b – as just combining two similar sets of sizes a and b. However, the procedure for "combining" is not representation independent. While simple "putting together" may work if we represent the numbers as collections of stones or marbles, it is not well defined for adding two numbers in the place-value representation that we are familiar with from junior school. Hence "combining" is a somewhat unsatisfactory way of defining addition.

A better way of defining $a + b$ is by using the successor operation $S(a) = a + 1$, b times. As long as we have a primitive method for computing $a + 1$ in any representation for an arbitrary a, this definition of $a + b$ becomes representation independent. We may define the basic property of addition using counting as:

For all $m, n \in \mathbb{N}$:

1. $m + 0 = m$
2. $m + S(n) = S(m + n)$

[1] 0 is usually not included in the Set of Natural numbers in Mathematics. We will however include 0 in the set of Natural numbers in this course. After all, it is quite natural to score a 0 in an examination

In the above definition we have used the same trick as in definition of the set $\mathbb{N}$ above, of defining a larger concept as a successor of a smaller concept. The process repeats, and the actual additions happen in the return path. For example,

$$
\begin{aligned}
&7+5\\
&\quad=(7+4)+1\\
&\quad=((7+3)+1)+1\\
&\quad=(((7+2)+1)+1)+1\\
&\quad=((((7+1)+1)+1)+1)+1\\
&\quad=(((((7+0)+1)+1)+1)+1)+1\\
&\quad=((((7+1)+1)+1)+1)+1\\
&\quad=(((8+1)+1)+1)+1\\
&\quad=((9+1)+1)+1\\
&\quad=(10+1)+1\\
&\quad=11+1\\
&\quad=12
\end{aligned}
$$

Note that the repeated substitution of a larger problem with a smaller problem is bounded, because the first condition of the definition works as a sentinel that we are bound to encounter as we keep reducing n.

We can then describe a procedure for computing $a+b$ (Algorithm 1) based on the above principle, but avoiding the deferred computations. The procedure takes a and b as input and returns *sum* as the output. $sum \leftarrow sum+1$ denotes the operation "*sum* is *assigned* $sum+1$" indicating that *sum* is incremented by 1.

Algorithm 1 An algorithm for $a+b$ by $+1$ b-times.

```
1: procedure ADD(a, b)
2:     counter ← 0
3:     sum ← a
4:     while counter < b do
5:         sum ← sum + 1
6:         counter ← counter + 1
       return sum
```

Exercise 2.2 *1. Assuming that the operation $a+1$ is available as a primitive, convince yourself that the above procedure for adding two numbers are correct.*

2. Argue that if the operation $a+1$ is available as a primitive, then the above algorithm for addition is representation independent.

3. Describe how the algorithm may be implemented using pebbles or marbles to represent numbers.

2.3.2 Multiplication

We can now define multiplication as repeated additions:

1. $n \times 0 = 0$, for all $n \in \mathbb{N}$
2. $n \times S(m) = n \times m + n$, for all $n, m \in \mathbb{N}$

Note that here again we have defined $n \times S(m)$, in terms of a smaller problem $m \times n$ of the same type.

Exercise 2.3 *1. Convince yourself that according to the above definition* $n \times m = \underbrace{n + n + n + \ldots + n}_{m \text{ times}}$.

2. Provide a representation independent algorithm, using only the successor function and addition, for multiplication of two numbers.

3. Describe how the algorithm may be implemented using pebbles or marbles to represent numbers.

2.3.3 Subtraction

To define the subtraction operation $m - n$, we may first define a predecessor operation $P(n)$ – analogous to $S(n)$ – as

1. $P(0)$ is undefined
2. $P(n) = n - 1$ for all $n > 0$.

We assume, as before, that we have a primitive counting based procedure for computing $P(n) = n - 1$ in any representation. We can define the subtraction operation $m - n$ similarly to addition:

For all $m, n \in \mathbb{N}, m \geq n$

1. $m - m = 0$
2. $m - n = S(P(m) - n)$

As before, note that $P(m) - n$ is a smaller problem that $m - n$.

The subtraction algorithm may then be given as:

Algorithm 2 An algorithm for $a - b$, $a \geq b$ by -1 b-times.

```
1: procedure SUBTRACT(a, b)
2:     counter ← 0
3:     while counter < b do
4:         a ← a − 1
5:         counter ← counter + 1
       return a
```

Exercise 2.4 *Provide alternative versions of Algorithms 1 and 2 without using the counter. Instead decrement* b *using* $b \leftarrow b - 1$ *repeatedly till* $b = 0$.

2.3.4 Division

Division is a natural requirement in civilised societies, mainly for sharing. However, it may not always be possible to divide natural numbers in equal proportions. For example, a collection of size 3 cannot be divided in two proportions of equal sizes without breaking up at least one member element. We have the *division theorem*:

Theorem 2.1 *Given two numbers* $a, b \in \mathbb{N}$*, there exist unique* $q, r \in \mathbb{N}$ *(quotient and remainder, respectively) such that* $a = bq + r$ *and* $0 \leq r < b$.

Proof: Let us first argue that such q and r exist. Repeatedly compute $a - b, a - 2b, a - 3b, \ldots, a - kb$, $k \geq 0$, till $a - kb < b$ and subtraction is possible no more. Set $q = k$ and $r = a - kb$. Clearly, q is the total number of times b can be subtracted from a, and $0 \leq r < b$. If $r = 0$ then b divides a exactly.

To argue that that q and r obtained by the above procedure are unique, let us suppose they are not. Then, there exist q_1, r_1 and q_2, r_2 such that

$$a = bq_1 + r_1, 0 \leq r_1 < 0$$
$$a = bq_2 + r_2, 0 \leq r_2 < 0$$

Without loss of generality, let us assume that $q_1 \geq q2$. The above implies that $b(q_1 - q_2) = r_2 - r_1$. One of two cases arise:

1. $q_1 = q_2$. This implies that $r_1 = r_2$, and hence uniqueness.
2. $q_1 > q_2$. This implies that $q_1 - q_2 \geq 1 \in \mathbb{N}$. Hence $r_2 - r_1 \geq b$. But this is not possible because $0 \leq r_1, r_2 < b$. Hence q and r must be unique.

□

In the above proof, we used *explicit construction* as a proof technique for establishing existence of such q and r, and *contradiction* for establishing their uniqueness. We will revisit these techniques later in the course when we discuss proofs.

Exercise 2.5 *Describe an algorithm using repeated subtraction that computes q and r given a and b.*

2.4 The Sets of Integers

We defined the subtraction operation $m - n, m \geq n, m, n \in \mathbb{N}$ as the number of times the successor operation $S()$ needs to be applied to reach m from n. This definition requires the restriction that $m \geq n$. An obvious generalisation is to remove the restriction and measure the difference in terms of either the successor $S()$ or the predecessor $P()$ operator. Subtraction then becomes directional, and we require negative numbers to represent the direction. This leads us to the set of integers

$$\mathbb{Z} = \{\ldots, -3, -2, -1, 0, 1, 2, \ldots\}$$

Arithmetic in the set $\mathbb{Z}$ follows the same principles as in $\mathbb{N}$, except that they are now directional.

Exercise 2.6 *Rework the definitions and the algorithms for addition, multiplication, subtraction and division in $\mathbb{Z}$.*

2.5 The Sets of Rationals

The division theorem tells us that given $m, n \in \mathbb{N}$, there exist $q, r \in \mathbb{N}$, such that m can be divided in to q parts of size n, possibly leaving a remainder $0 \leq r < n$. Division is an obvious fundamental need for resource sharing. If each unit is indivisible – like live cattle, for example – then the division theorem is the best we can do. However, items measured in units such as weight, volume or length – such as meat from a hunted animal, or a pile of grains – are often divisible in smaller proportions like $1/3^{rd}$, $2/25^{th}$ etc. So, division inevitably leads us to fractions. We define the set of Rational numbers as

$$\mathbb{Q} = \{x | x = p/q, p \in \mathbb{Z}, q \in \mathbb{N}, q \neq 0\}$$

We often also write this as $\frac{p}{q}$. These are numbers of the type $\pm 1/1, \pm 1/2, \pm 1/3, \pm 1/4, \pm 2/5$ etc. We may also insist that p and q should have no common factors (i.e., $gcd(p, q) = 1$; see Section 3.5 for a formal definition of *gcd*) to avoid multiple representations for the same Rational number. Clearly $\mathbb{N} \subset \mathbb{Z} \subset \mathbb{Q}$.

Note, however, that we now have a situation where between any two rational numbers there are infinitely many other rational numbers.

Exercise 2.7 *Convince yourself of the above statement.*

This implies that there is no well defined successor function for a rational number, and we need to revisit our definition of addition for rationals. We start by noting that, for example,

$$\frac{1}{3} + \frac{2}{3} = \frac{1+2}{3} = 1$$

i.e., we can add the numerators as in $\mathbb{Z}$ if the denominators are the same. However, the addition

$$\frac{2}{3} + \frac{3}{4}$$

is not well defined unless the two fractions can be expressed in the same unit. But we can multiply the numerator and denominator of the first fraction by 4, and the second by three to convert to the same unit where the denominator of both is 12

$$\frac{2\times 4}{3\times 4}+\frac{3\times 3}{4\times 3}=\frac{8}{12}+\frac{9}{12}=\frac{8+9}{12}=\frac{17}{12}$$

Note that multiplying the numerator and the denominator of a fraction with the same number does not change the fraction. So, we can define the general rule for addition of two rational numbers as

$$\frac{p_1}{q_1}+\frac{p_2}{q_2}=\frac{p_1\times q_2+p_2\times q_1}{q_1\times q_2}$$

where all the additions and multiplications are defined on the set $\mathbb{Z}$.

Exercise 2.8 *Extend the above idea to define subtraction, multiplication and division in the set* $\mathbb{Q}$.

Problems

1. Give three different representations for the number 6 (for example: tally marks, Roman numerals, line segment lengths). Explain how the operation "+1" is carried out in each representation.
2. Research and briefly describe how numbers were represented in one historical number system not discussed in class (e.g., Mayan or Incan). Compare it with the decimal positional system.
3. Let

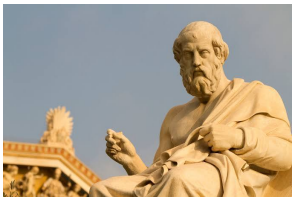
   $$A=\{x \mid x \text{ is an even number less than } 20\},\quad B=\{x \mid x \text{ is a prime number less than } 20\}.$$

   Compute $A\cap B$, $A\cup B$, and $A\setminus B$.
4. Prove or disprove: If $A\subseteq B$ and $B\subseteq C$, then $A\subseteq C$.
5. Construct a real-world example of three sets A, B, C such that $C\subset B\subset A$.
6. Using only the successor function $S(n)=n+1$, show step by step how to compute $4+3$.
7. Show that addition defined by the recursive rules in the text is associative, i.e. prove $(a+b)+c=a+(b+c)$ using the definition.
8. Starting from the definition $n\times S(m)=(n\times m)+n$, compute 3×4 step by step.
9. Write pseudocode (representation-independent) for multiplication using only the successor function and addition.
10. Explain how you could simulate multiplication using only pebbles to represent numbers.
11. Using the predecessor function $P(n)$, compute $7-4$ step by step as in the recursive definition.
12. Implement Algorithm 2 (subtraction by repeated decrementing of b) on the input $a=10, b=6$. Show the intermediate steps.
13. Using the Division Theorem, compute the quotient and remainder when $a=29, b=5$ using repeated subtraction.
14. Prove that the quotient and remainder obtained from the Division Theorem are unique.
15. Extend the recursive definition of addition from natural numbers to integers, and compute $(-3)+5$.
16. Explain why we need negative numbers to generalize subtraction. Give a real-world example where negative numbers are essential.

17. Give an example of two distinct rational numbers between $\frac{1}{3}$ and $\frac{1}{2}$.
18. Prove that between any two rational numbers there exists another rational number. (Hint: use their average.)
19. Compute $\frac{2}{3} + \frac{4}{5}$ using the common-denominator method.
20. Extend the definition to show how to compute $\frac{3}{7} \div \frac{2}{5}$.

Chapter 3

Ruler and compass algorithms



"Geometry is knowledge of the eternally existent... it compels the soul to look upwards, and leads us away from the world of appearance to the vision of truth."
"Let no one ignorant of geometry enter here" – Plato

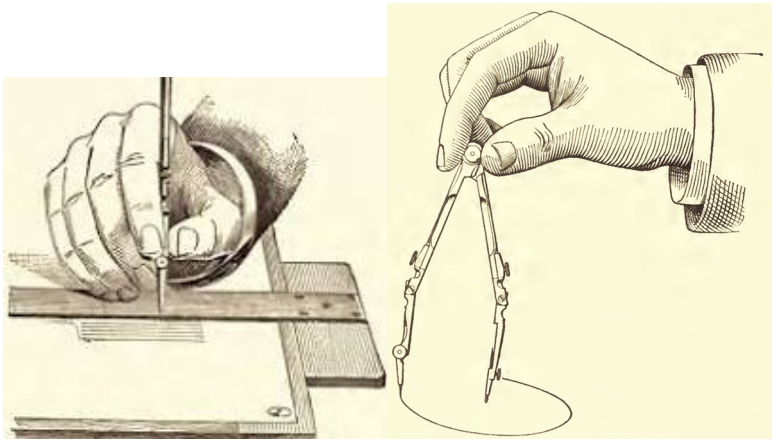


Figure 3.1: Ruler and compass. We will assume that the ruler is unmarked, and lengths can only be measured by adjusting the width of the compass.

Our endeavour so far has been to define numbers, and operations on them, in a representation independent manner. Let us now consider a specific computational model – straightedge and compass constructions introduced by the ancient Greeks – and examine whether the abstract operations we have defined above can be translated in to definite constructible procedures, or *algorithms*. Most of the geometric constructions date back to Euclid's books of Elements from around 300 BCE. We will often – by force of habit – refer to them as *ruler and compass constructions* but with the understanding that the ruler has no markings for length measurements, and can only be used to draw straight edges. See Figure 3.1.

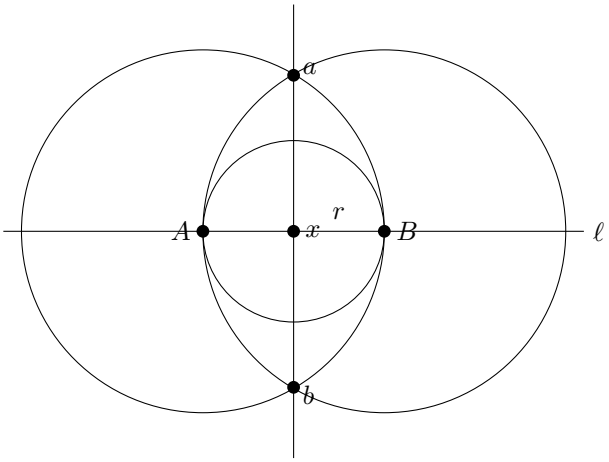


Figure 3.2: Constructing a line perpendicular to a given line passing through a point

Let us first consider some basic constructions. In what follows, the correctness of the constructions will often rely on the basic geometric properties of similar triangles. A reader may revise them from here and here.

3.1 Constructing a line perpendicular to a given line passing through a point

Given a straight line ℓ, and a point x on it, let us consider the problem of constructing a line perpendicular to ℓ and passing through x. We give a construction in Algorithm 3 (See Figure 3.2).

Algorithm 3 Constructing a line perpendicular to a given line passing through a point

1: **procedure** PERPENDICULAR(x, ℓ)
2: $c = CIRCLE(x, r)$, where r is a random length
3: $(A, B) = c \cap \ell$
4: $c_A = CIRCLE(A, 2r)$
5: $c_B = CIRCLE(B, 2r)$
6: $(a, b) = c_A \cap c_B$
7: $result = LINE(a, b)$

We have described the algorithmic procedure using some standard primitives. $c = CIRCLE(x, r)$ denotes the construction of a circle c centred at x of radius r. A and B are the intersection points of c with ℓ, denoted in the algorithm as $(A, B) = c \cap \ell$. Similarly, c_A and c_B are circles of radius $2r$ centred at A and B respectively, and a and b are the intersection points of c_A and c_B. $LINE(a, b)$ joins a and b and is the *result*.

Exercise 3.1 *1. Convince yourself that the above construction is correct. Use properties of similar triangles.*

2. Argue that $LINE(a, b)$ is also the perpendicular bisector of AB.

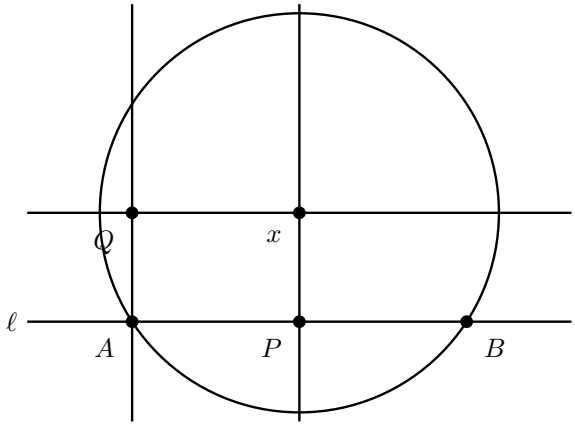


Figure 3.3: Constructing a line parallel to a given line passing through a point

3.2 Constructing a line parallel to a given line passing through a point

Given a line ℓ, and an arbitrary point x, consider the problem of construction of a line parallel to ℓ passing through x. We give a construction in Algorithm 4 (See Figure 3.3).

Algorithm 4 Constructing a line parallel to a given line passing through a point

1: **procedure** PARALLEL(x, ℓ)
2: $c = CIRCLE(x, r)$, where r is a random length
3: $(A, B) = c \cap \ell$
4: Construct p, the perpendicular bisector of AB using Algorithm 3. Argue that p passes through x.
5: $P = p \cap \ell$
6: Construct q, a perpendicular to ℓ passing through A using Algorithm 3.
7: Construct s, a perpendicular to p passing through x using Algorithm 3.
8: $Q = q \cap s$
9: $result = LINE(Q, x)$

Note that in steps 4,6 and 7 of the algorithm, we have used the procedure of Algorithm 3. We will routinely use a previously defined algorithm as a primitive to define a new algorithm.

3.3 Constructibility and the compass equivalence theorem

The above two sections give us several examples of construction of points, lines, line segments and circles. The informal definition of *constructibility* is as follows. Given points are by definition constructible. A line joining two constructible points is constructible. So is the circle centred on one constructible point passing through another constructible point. A point is constructible if it is an intersection of constructible lines and circles.

The compass advocated by the Greek philosopher Plato in these constructions is a *collapsing compass*, i.e., a compass that "collapses" whenever it is lifted from a page, so that it may not be directly used to transfer distances unlike in a modern fixable aperture compass. Note that nowhere in Sections 3.1 and 3.2 have we transferred distances using a fixed size compass lifted from the page.

This is however not a limitation as the following construction shows.

Theorem 3.1 *A collapsing compass can be used to transfer a given length to an arbitrary given point*

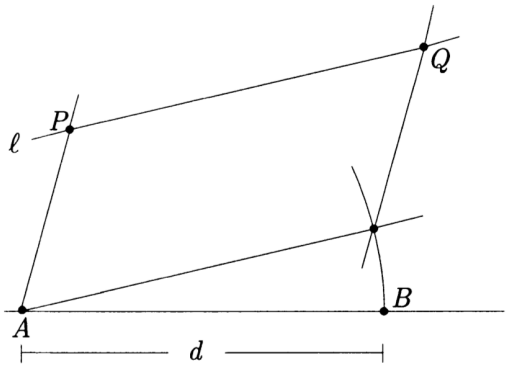


Figure 3.4: A compass-equivalence construction

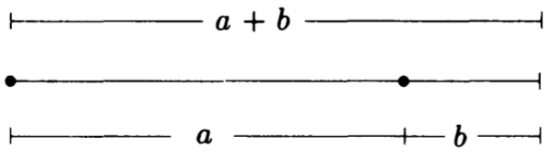


Figure 3.5: Construction of $a + b$.

on a given line. That is, given an arbitrary length $\overline{AB}$, a line ℓ and a point P on it, it is possible to construct using a collapsing compass a point Q on ℓ such that $\overline{PQ} = \overline{AB}$.

Proof:

Algorithm 5 A compass-equivalence construction

1: **procedure** COMPASS-EQUIVALENCE(A,B,ℓ,P)
2: $c = CIRCLE(A, B)$; note that $\overline{AB} = d$
3: Construct p, the line parallel to ℓ passing through A using Algorithm 4
4: $R = c \cap p$
5: $q = LINE(A, P)$
6: Construct s, the line parallel to q passing through R using Algorithm 4
7: $Q = s \cap \ell$
8: $\overline{PQ}$ is the result on ℓ

Exercise 3.2 *Convince yourself that $\overline{PQ} = \overline{AB}$.*

□

The above is different from the original construction and proof of correctness as given by Euclid in his Book of Elements. Interested readers may study a modern version of the original construction here.

3.4 Rational numbers are constructible

First, we can consider any given length as 1, measure it using a non-collapsing compass – which we just proved is equivalent to a collapsing compass – and add it to any point on a given line by marking it off with the compass. We can similarly subtract. The operations $S(n)$ and $P(n)$ are thus

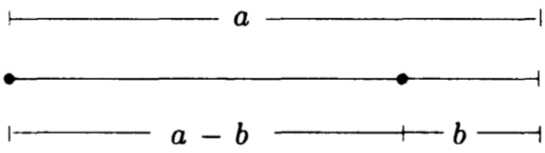


Figure 3.6: Construction of $a - b$.

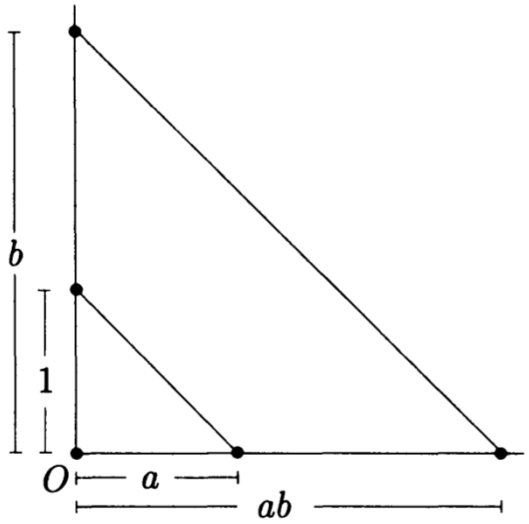


Figure 3.7: Construction of ab given a and b.

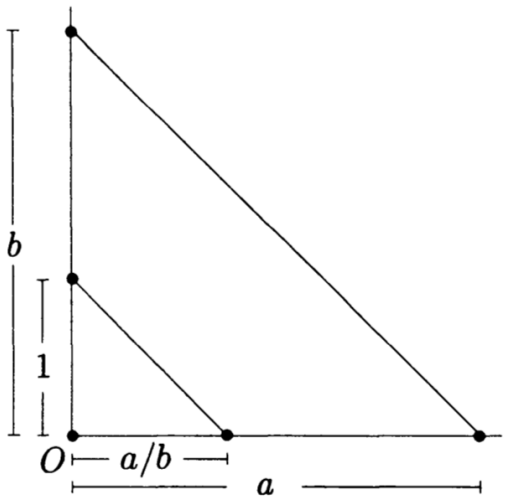


Figure 3.8: Construction of a/b given a and b.

realisable using ruler and compass. Consequently, the elements of the set $\mathbb{Z}$ are constructible. In Figures 3.5 and 3.6 we give the direct constructions for $a+b$ and $a-b$.

Exercise 3.3 *1. Describe a ruler and compass procedure for multiplication using repeated additions.*

2. Describe a ruler and compass procedure for division (computing quotient and remainder) using repeated subtractions.

Given integers a and b as line segments, we can also construct rational number ab and a/b directly using similar triangles. The construction of Figure 3.7 involves marking off the lengths a and b in two perpendicular segments from O, constructing the unit length in the direction of b, and constructing a line parallel to the line $\overline{a1}$ through a or b. The intercept of the parallel line in the direction of a then marks the length ab by similarity of the triangles.

We can similarly construct a/b as depicted in Figure 3.8. Rational numbers are thus constructible.

3.5 Euclid's GCD using ruler and compass

GCD of two integers $a > 0$, $b \geq 0$ is defined as the largest integer $d, d > 0$ that divides both a and b. Consider the following algorithm for computing the GCD:

$$gcd(a,b) = \begin{cases} a & \text{if } b = 0 \\ a & \text{if } a = b \\ gcd(a-b,b) & \text{if } a > b \\ gcd(a,b-a) & \text{if } b > a \end{cases}$$

Exercise 3.4 *1. Convince yourself that the above algorithmic specification (rule) is correct for computing GCD. Carry out the pencil and paper computation using the above algorithm for some special cases.*

2. Describe the procedure for executing the algorithm using ruler and compass.

The algorithm described above is from Euclid's Elements. You can find a description of it here. This is also considered to be the oldest non-trivial algorithm in common use.

Now that we have defined our first computational model, several questions arise. What are the full powers of the model? What are the other things that can be constructed? What are the limits of the model, and are there easily defied concepts that are not constructible? Can there be other computational models more powerful than ruler and compass? These are the kind of questions we interrogate every computational model with. We will revisit some of these questions in the latter chapters.

Exercise 3.5 *Try to construct $\sqrt{2}$ using ruler and compass. (Hint: Use the Pythagoras theorem on a right triangle with sides $1, 1$.) Argue why this length is constructible.*

Problems

1. Construct, using ruler and compass, the perpendicular bisector of a line segment AB. Show that any point P on the perpendicular bisector is equidistant from A and B.

2. Given a triangle ABC, construct the three medians using ruler and compass. Argue that they intersect at a single point (the centroid).

3. Using Algorithm 3 (perpendicular construction), describe step by step how to construct the altitude from a vertex of a triangle. Illustrate your steps with a figure.

4. Given a line ℓ and an external point P, use Algorithm 4 (parallel construction) to draw the line parallel to ℓ through P. Prove that the two lines do not intersect.

5. Show that the compass-equivalence construction (Algorithm 5) indeed allows one to transfer a given length AB to an arbitrary point P on a line ℓ. Verify your construction with a worked example.

6. Describe a ruler and compass procedure for multiplication of two integers a, b using repeated additions. Demonstrate the construction for $a = 3, b = 4$.

7. Similarly, describe a ruler and compass procedure for division using repeated subtractions. Apply your method to compute the quotient and remainder when dividing a segment of length 11 into parts of length 3.

8. Using constructions based on similar triangles, show step by step how to obtain the product ab given line segments a and b.

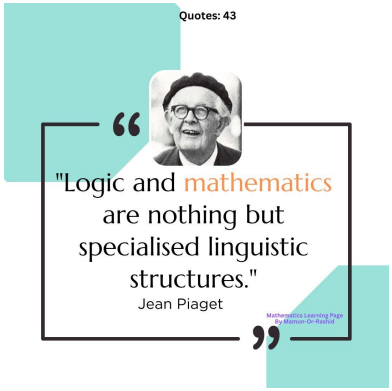
9. Construct the rational number $\frac{3}{5}$ on a line, starting with a unit segment. Explain each step of your construction.

10. Apply Euclid's GCD algorithm (as given in Section 3.5) to the lengths $a = 21$, $b = 15$, using repeated subtraction with compass and ruler. Show all intermediate steps.

11. Prove that Euclid's GCD algorithm terminates in a finite number of steps for all $a, b \in \mathbb{N}$. (Hint: in each step one of the arguments strictly decreases.)

Chapter 4

Abstraction turns problems and concepts into principles



Modern mathematics and computer science are built upon a few simple but very powerful ideas. Among the most important are the notions of *relations* and *functions*, which allow us to describe how objects are connected or transformed. Counting, infinity, and the ways in which sets can be grouped into classes help us measure size, structure, and complexity. These ideas may look abstract at first, but they underlie the methods used in algorithms, data structures, and logical reasoning.

In computer science, functions capture the essence of computation: a program takes inputs and produces outputs, just as a function does. Understanding one-one and onto functions helps us reason about whether information is lost, preserved, or fully covered. Equivalence classes and partitions allow us to organise data into categories. Modular arithmetic, often called "clock arithmetic," is fundamental in cryptography, error detection, and digital systems. Learning these concepts gives us a foundation to explore deeper mathematics and to apply it to practical computational problems.

4.1 Relations

The *Cartesian product* of two sets A and B, denoted by $A \times B$, is the set of all ordered pairs (a, b) such that $a \in A$ and $b \in B$. Thus,

$$A \times B = \{(a, b) \mid (a \in A) \text{ and } (b \in B)\}$$

A^n is the set of all ordered n-tuples $(a_1, a_2, \ldots, a_n)$ such that $a_i \in A$ for all i. i.e.,

$$A^n = \underbrace{A \times A \times \cdots \times A}_{n \text{ times}}$$

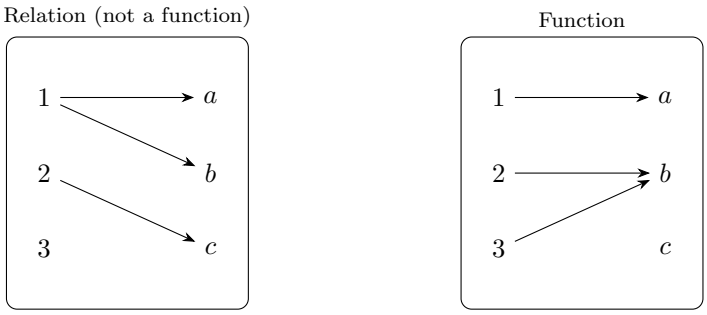


Figure 4.1: Relation vs function: In a relation, an element in A can be mapped to multiple elements in B, but not so in a function. In a function, all elements in A must be covered, but not necessarily so in a relation. The set B need not be fully covered in either.

A *relation* tells us which elements of one set are connected to elements of another.

A *binary relation* R from A to B is a subset of $A \times B$. It is a characterisation of the intuitive notion that some of the elements of A are related to some of the elements of B.

If $A = \{1, 2, 3\}$ and $B = \{a, b, c\}$, then $R = \{(1, a), (2, b), (3, a)\}$ is a relation from A to B. Familiar binary relations from $\mathbb{N}$ to $\mathbb{N}$ are $=$, $\neq$, $<$, $\leq$, $>$, $\geq$. Thus the elements of the set $\{(0, 0), (0, 1), (0, 2), \ldots, (1, 1), (1, 2), \ldots\}$ are all members of the relation $\leq$ which is a subset of $\mathbb{N} \times \mathbb{N}$.

4.2 Function

A *function* from A to B – written as $f : A \to B$ – is a special relation in which:

1. every element of A is related to some element of B, and
2. no element of A is related to more than one element of B.

Equivalently, each input has *exactly one* output.

Some familiar examples of functions are

1. $+$ and $*$ (addition and multiplication) are functions of the type $f : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$
2. $-$ (subtraction) is a function of the type $f : \mathbb{N} \times \mathbb{N} \to \mathbb{Z}$.
3. *div* and *mod* are functions of the type $f : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$. If $a = q * b + r$ such that $0 \leq r < b$ and $a, b, q, r \in \mathbb{N}$ then the functions *div* and *mod* are defined as $div(a, b) = q$ and $mod(a, b) = r$. We will often write these binary functions as $a * b$, a *div* b, a *mod* b etc.
4. The binary relations $=$, $\neq$, $<$, $\leq$, $>$, $\geq$ are also functions of the type $f : \mathbb{N} \times \mathbb{N} \to \mathbb{B}$ where $\mathbb{B} = \{false, true\}$.
5. $f : \mathbb{N} \to \mathbb{N},\ f(x) = x^2$.

We write the definition of a function formally as follows.

A *function* from A to B is a binary relation f from A to B such that for *every* element $a \in A$ there is a *unique* element $b \in B$ so the $(a, b) \in f$ (or $f(a) = b$)[1]. We will use the notation $f : A \to B$ to denote a function f from A to B. The set A is called the *domain* of the function f and the set B is called the *co-domain* of the function f. The *range* of a function $f : A \to B$ is the set $\{b \in B \mid \text{for some } a \in A,\ f(a) = b\}$ denoting the subset of elements in B that are actually covered by f.

[1]This is sometimes written using mathematical notation as $\forall a \in A$, $\exists$ unique $b \in B$. $\forall$ is the usual symbol for *for all*, and $\exists$ is the usual symbol for *there exists*

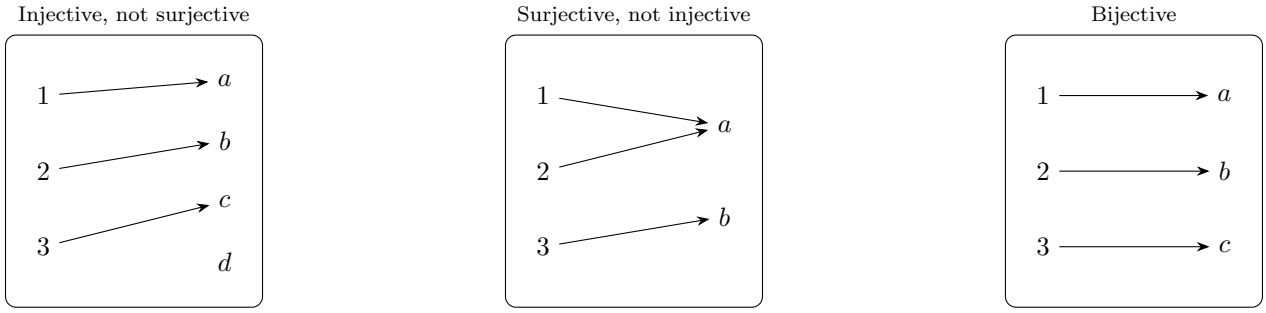


Figure 4.2: *One-one (injective)*: different inputs give different outputs. (like roll numbers: no two students share one roll); *Onto (surjective)*: every element of the codomain gets hit by *some* input. (no empty seats.); *Bijective*: both one-one and onto. (perfect pairing; has an inverse.)

4.2.1 One-One (injective), Onto (surjective), and bijective Functions

When we study functions, it is often not enough to know that "every input has exactly one output." We also want to understand how well the function uses its codomain and whether different inputs remain distinct after applying the function.

Think of a classroom with students and seats:

- If no two students sit in the same seat, the "assignment" of students to seats is one-one (injective).
- If every seat is occupied by at least one student, the assignment is onto (surjective).
- If both conditions happen together — each student has exactly one seat, and every seat is filled — then the assignment is bijective. In such a case we may also define an inverse function from seats to students.

The formal definitions below make the concepts precise.

Let $f : A \to B$ be a function. f is

Injective if whenever $f(a_1) = f(a_2)$ for $a_1, a_2 \in A$, we can conclude that $a_1 = a_2$

Surjective if for all $b \in B$, there exists $a \in A$ such that $f(a) = b$.

Bijective if it is both injective and surjective.

Example 4.1 *1.* $f : \mathbb{N} \to \mathbb{R},\ f(n) = 2n$ *is injective but not surjective (odd numbers are not covered).*

2. $g : \mathbb{N} \to \mathbb{N},\ g(0) = 0;\ g(n) = n - 1$ *is surjective but not injective (why?).*

4.3 Counting, Finite and Infinite Sets

Counting is one of the most fundamental activities in mathematics: it is how we measure the size of a set. At first glance this seems very straightforward — counting laddoos in a box or students in a class is familiar to everyone. However, mathematics asks us to extend this simple idea to more abstract settings: What does it mean for a set to be "finite"? How can we formally compare the sizes of different sets, especially when some sets are infinite? For finite sets, the answer is clear: we can match the elements of the set to the numbers $\{1, 2, \ldots, n\}$. For infinite sets, the situation is more subtle, but surprisingly we can still talk about "countable" sets — those whose elements can be placed in a one-to-one correspondence with the natural numbers $\mathbb{N}$. This point of view leads to some striking and important results: for instance, that the set of all even numbers, the set of all

integers, and even the set of all rational numbers are all countably infinite. At the same time, we will see later in this course that there are sets of numbers so large that they cannot even be listed in sequence: these are called uncountable sets. These distinctions between finite, countably infinite, and uncountable sets form the foundation for much of modern mathematics, and are crucial in computer science as well, where questions of size, encoding, and enumeration play a central role.

4.3.1 Finite sets

A set is *finite* if it has a finite number of elements. Formally, a set A is finite if there exists a natural number n and a bijection

$$f : A \to \{1, 2, \ldots, n\}.$$

This means that the elements of A can be paired exactly with the first n natural numbers.

Example 4.2 *The set $\{a, b, c\}$ is finite because we can define $f(a) = 1$, $f(b) = 2$, $f(c) = 3$, which is a bijection to $\{1, 2, 3\}$.*

4.3.2 Infinite sets and bijections to $\mathbb{N}$

A set is *infinite* if it is not finite. Some infinite sets are still "countable" because they can be put in one-to-one correspondence (bijection) with the natural numbers $\mathbb{N}$. In such a case the elements of the set can be enumerated as *first*, *second*. *third*, and so on.

Definition. A set A is *countably infinite* or *denumerable* if there exists a bijection $f : A \to \mathbb{N}$.

Example 4.3 *1. The natural numbers $\mathbb{N}$ are countably infinite via the trivial bijection $f(n) = n$.*

2. *The set of even naturals $E = \{0, 2, 4, 6, \ldots\}$ is countably infinite. Define $f : \mathbb{N} \to E$ by $f(n) = 2n$. This is a bijection.*

3. *The set of odd naturals $O = \{1, 3, 5, 7, \ldots\}$ is also countably infinite. Define $g : \mathbb{N} \to O$ by $g(n) = 2n + 1$. This is a bijection.*

4.3.3 Integers and Rationals are countable

The integers $\mathbb{Z} = \{\ldots, -2, -1, 0, 1, 2, \ldots\}$ can be enumerated in a sequence:

$$0, -1, 1, -2, 2, -3, 3, -4, 4, \ldots$$

Define a bijection $h : \mathbb{N} \to \mathbb{Z}$ by

$$h(n) = \begin{cases} \frac{n}{2} & \text{if } n \text{ is even,} \\ -\frac{n+1}{2} & \text{if } n \text{ is odd.} \end{cases}$$

This shows that $\mathbb{Z}$ is countable.

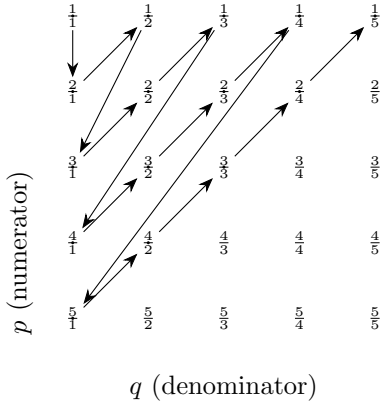
The rationals $\mathbb{Q} = \{\frac{p}{q} : p, q \in \mathbb{Z}, q \neq 0\}$ are also countable, though the proof is less obvious.

Step 1. First, consider only the positive rationals $\mathbb{Q}^+$. We can arrange them in a grid with numerator along one axis and denominator along the other:

	1	2	3	...
1	$\frac{1}{1}$	$\frac{1}{2}$	$\frac{1}{3}$	...
2	$\frac{2}{1}$	$\frac{2}{2}$	$\frac{2}{3}$	...
3	$\frac{3}{1}$	$\frac{3}{2}$	$\frac{3}{3}$	...
⋮	⋮	⋮	⋮	⋱

Note, however, that this does not give us an enumeration.

Step 2. Use Cantor's method: start at $\frac{1}{1}$, then $\frac{2}{1}, \frac{1}{2}$, then $\frac{3}{1}, \frac{2}{2}, \frac{1}{3}$, and so on, zig-zagging across the grid. This produces a sequence that eventually lists every positive rational.



Exercise 4.1 *1. Convince yourself that above ordering gives a bijection from* $\mathbb{N}$ *to ordered pairs* $(p, q), p > 0, q > 0$.

*2. Can you work out an explicit formula for the bijective function? (***This can be challenging***)*

Step 3. To avoid repetitions, we can restrict to fractions in lowest terms (e.g. $\frac{2}{2}$ is skipped since it equals $\frac{1}{1}$).

Step 4. To cover negative rationals as well, interleave them with positives:

$$0, \tfrac{1}{1}, -\tfrac{1}{1}, \tfrac{2}{1}, -\tfrac{2}{1}, \tfrac{1}{2}, -\tfrac{1}{2}, \ldots$$

This construction defines an explicit enumeration of $\mathbb{Q}$, so $\mathbb{Q}$ is countably infinite.

4.4 Equivalence Relations, Classes, and Partitions

In mathematics, we often want to group objects together when they share some common property. For example, in geometry all shapes that have the same size and shape are considered "congruent," and in number theory two integers that leave the same remainder when divided by n are considered "equivalent." These situations are captured formally by the concept of an *equivalence relation*.

Equivalence relations are important because they let us partition a large and possibly complicated set into smaller, simpler pieces (called *equivalence classes*). Each equivalence class collects all elements that are considered "the same" under the relation. Many areas of mathematics, and even computer science (e.g. hashing, classification, state-space reduction), rely on such partitions.

Definition. A relation R on a set A is an *equivalence relation* if for all $a, b, c \in A$:

- (Reflexive) $(a, a) \in R$,
- (Symmetric) $(a, b) \in R \Rightarrow (b, a) \in R$,
- (Transitive) $(a, b) \in R$ and $(b, c) \in R \Rightarrow (a, c) \in R$.

We will also write $(a, b) \in R$ as $a \; R \; b$ or as $a \sim b$.

Definition. Set of elements that are equivalent form an equivalent class. Given $a \in A$, the *equivalence class* of a under relation R is

$$[a] = \{x \in A : (a, x) \in R\}.$$

Here are some example of equivalent relations

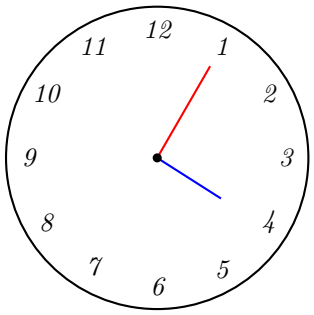
Example 4.4 1. ***Same birthday.*** *On the set of all students, define $x \sim y$ if x and y have the same birthday. Each equivalence class is a group of students born on the same day.*

2. ***Congruent triangles.*** *On the set of all triangles in the plane, define $T_1 \sim T_2$ if T_1 and T_2 are congruent (same shape and size). This relation is reflexive (every triangle is congruent to itself), symmetric (if T_1 is congruent to T_2, then T_2 is congruent to T_1), and transitive (if T_1 is congruent to T_2 and T_2 to T_3, then T_1 is congruent to T_3).*

3. ***Sets with same cardinality.*** *Clearly, $A \sim B$ if there exists a bijection between A and B. Since bijective functions have inverses that are bijections, and composition of two bijective functions is a bijection, we have that sets with same cardinality for an equivalent class.*

4. ***Congruence of integers (mod n).*** *Define $a \sim b$ if n divides $a - b$. We write this as $a \sim b \pmod n$ or $a \equiv b \pmod n$. This is reflexive ($a - a = 0$ is divisible by n), symmetric (if $a - b$ divisible by n, so is $b - a$), and transitive (if $a - b$ and $b - c$ divisible by n, so is $a - c$). The equivalence classes are the sets of integers with the same remainder when divided by n.*

   For example, on a 12-hour clock, $16 \sim 4 \pmod{12}$; and, for the minutes hand, $65 \sim 5 \pmod{60}$;



Exercise 4.2 *Which of the following relations are not equivalent relations and why?*

1. *a R b if $a = b$.*
2. *a R b if $a \leq b$.*
3. *a R b if $gcd(a, b) = 1$.*

4.4.1 Equivalence classes and partitions

Theorem 4.1 *An equivalence relation defined on set A partitions A into disjoint equivalence classes: every element of A belongs to exactly one equivalence class, and the classes together cover all of A.*

Proof: We have to argue for two things – first, no element belongs to two or more equivalent classes; and second, the union of all the equivalence classes covers the whole set.

Suppose x belongs to two distinct equivalence classes. Then there exits $a, b \in A$ such that $x \sim a$ and $x \sim b$ but $a \not\sim b$. But this is not possible because $\sim$ is symmetric and transitive.

And, by reflexivity, each x is at least equivalent to itself. So, no element is left out. □

Example 4.5 *The relation Congruence of integers (mod 3) split $\mathbb{Z}$ into three classes:*

$$[0] = \{\ldots, -6, -3, 0, 3, 6, \ldots\}, \quad [1] = \{\ldots, -5, -2, 1, 4, 7, \ldots\}, \quad [2] = \{\ldots, -4, -1, 2, 5, 8, \ldots\}.$$

These three classes form a partition of $\mathbb{Z}$.

4.5 Modular Arithmetic, Magic Squares, and One-Time Pads

4.5.1 Modular arithmetic

Consider, for example, the set $\mathbb{Z}_7 = \{0, 1, 2, 3, 4, 5, 6\}$ consists of all integers modulo 7. This means that we can perform addition and multiplication, and then reduce the result to its remainder upon division by 7.

Addition. Examples:

$$3 + 5 \equiv 1 \pmod 7, \quad 6 + 4 \equiv 3 \pmod 7.$$

Every element has an *additive inverse*, i.e. a number x such that $a + x \equiv 0 \pmod 7$.

$$0^{-1} = 0, \quad 1^{-1} = 6, \quad 2^{-1} = 5, \quad 3^{-1} = 4.$$

Multiplication. Examples:

$$3 \times 5 \equiv 1 \pmod 7, \quad 4 \times 6 \equiv 3 \pmod 7.$$

For multiplication, every nonzero element has a *multiplicative inverse*, i.e. $a \cdot x \equiv 1 \pmod 7$.

$$1^{-1} = 1, \quad 2^{-1} = 4, \quad 3^{-1} = 5, \quad 6^{-1} = 6.$$

Notice that 0 has no multiplicative inverse.

Thus $(\mathbb{Z}_7, +)$ is a finite arithmetic structure under addition, and $(\mathbb{Z}_7^\times, \cdot)$ with $\{1, 2, 3, 4, 5, 6\}$ is a finite arithmetic structure under multiplication. This has many interesting applications.

4.5.2 Magic Squares

A *magic square* is an arrangement of numbers in a square grid such that the sums of each row, column, and both diagonals are the same. Modular arithmetic allows us to construct magic squares in structures like $\mathbb{Z}_n$, where the "magic sum" is computed modulo n. This is a playful illustration of modular addition applied to combinatorial design.

A simple way to build a magic square *modulo* n is to take any ordinary magic square and reduce each entry modulo n. The LuoShu 3×3 square, which has a rich history in occult and numerology is given as

4	9	2
3	5	7
8	1	6

(each line sums to 15)

It reduces modulo 7 to

4	2	2
3	5	0
1	1	6

$\in \mathbb{Z}_7^{3\times 3}$.

Since $15 \equiv 1 \pmod 7$, every row, column, and diagonal now sums to $\overline{1} \in \mathbb{Z}_7$. For instance,

$$4 + 2 + 2 \equiv 8 \equiv 1 \pmod 7, \qquad 3 + 5 + 0 \equiv 8 \equiv 1 \pmod 7, \qquad 1 + 5 + 2 \equiv 8 \equiv 1 \pmod 7.$$

The diagonals also satisfy $4 + 5 + 6 \equiv 15 \equiv 1 \pmod 7$ and $2 + 5 + 1 \equiv 8 \equiv 1 \pmod 7$.

General recipe (odd moduli). Let S be any 3×3 magic square over the integers with magic sum M. For any modulus $n \geq 2$, the entry-wise reduction $\overline{S} \in (\mathbb{Z}_n)^{3\times 3}$ is a magic square in $\mathbb{Z}_n$ with magic sum $\overline{M} \in \mathbb{Z}_n$, because modular addition preserves equality of sums. More generally, for any $\alpha, \beta \in \mathbb{Z}_n$, the entry-wise transform $\alpha S + \beta$ is again a magic square modulo n with magic sum $\alpha M + 3\beta \pmod n$.

4.5.3 Perfect Secrecy and One-Time Pads

The idea of modular arithmetic is also central in cryptography. In the *one-time pad*, a message (plaintext) is converted into numbers (say letters $A = 0, \ldots, Z = 25$). A random secret key of the same length is chosen, and encryption is done by addition modulo 26:

$$\text{ciphertext}_i \equiv \text{plaintext}_i + \text{key}_i \pmod{26}.$$

Decryption uses subtraction modulo 26.

Claude Shannon showed that if the key is truly random, used only once, and kept secret, then the ciphertext reveals no information about the plaintext: this is called *perfect secrecy*. Thus, a simple application of modular addition gives us a theoretically unbreakable cryptosystem.

Summary

Modular arithmetic provides the framework to work with remainders in a structured way. It underlies recreational mathematics like magic squares, as well as fundamental cryptographic protocols such as the one-time pad.

Problems

1. Let $A = \{1, 2, 3\}$ and $B = \{a, b, c\}$. (a) List all elements of the Cartesian product $A \times B$. (b) Define two different binary relations $R_1, R_2 \subseteq A \times B$. (c) Which of them, if any, are functions?

2. Show that the relation $\leq$ on $\mathbb{N}$ is a subset of $\mathbb{N} \times \mathbb{N}$. Explicitly write out the first ten elements of this relation.

3. Give an example of a relation from $\{1, 2, 3\}$ to $\{a, b\}$ that is not a function. Explain why it fails to satisfy the definition of a function.

4. Consider the function $f : \mathbb{N} \to \mathbb{N}$ defined by $f(n) = 2n$. Argue that f is injective but not surjective.

5. Consider the function $g : \mathbb{N} \to \mathbb{N}$ defined by

$$g(0) = 0, \qquad g(n) = n - 1 \text{ for } n > 0.$$

   Show that g is surjective but not injective.

6. Let $h : \mathbb{Z} \to \mathbb{Z}$ be defined by $h(x) = x + 5$. Prove that h is bijective and describe its inverse function.

7. Define the function $p : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ by $p(a, b) = a \times b$. Argue formally that p is a well-defined function.

8. Construct an example of a function $f : A \to B$ where $A = \{1, 2, 3, 4\}$, $B = \{a, b\}$ that is neither injective nor surjective. Explain why.

9. Prove that if a function $f : A \to B$ is bijective, then there exists a unique inverse function $f^{-1} : B \to A$ such that $f^{-1}(f(a)) = a$ for all $a \in A$.

10. Consider the function $f : \mathbb{N} \to \mathbb{N}$ defined recursively as

$$f(0) = 0, \quad f(n + 1) = f(n) + (2n + 1).$$

    Show that $f(n) = n^2$. What kind of function is this (injective, surjective, bijective)?

11. Give an everyday example (outside mathematics) of: (a) an injective mapping, (b) a surjective mapping, and (c) a bijective mapping.

12. Challenge: Prove or disprove — the composition of two injective functions is injective, and the composition of two surjective functions is surjective. What about bijective functions?

13. On the set $\{1, 2, 3, 4, 5, 6\}$, define a relation R by $a \sim b$ if $a - b$ is divisible by 2.

    (a) Show that R is an equivalence relation.

    (b) List the equivalence classes.

14. Consider the relation "x has the same number of letters as y" on the set of English words. Prove or disprove that it is an equivalence relation. What do the equivalence classes look like?

15. Work out the addition and multiplication tables of $\mathbb{Z}_5$. Identify all additive and multiplicative inverses.

16. In $\mathbb{Z}_7$, solve the linear congruence $3x \equiv 2 \pmod 7$.

17. Find all solutions to $x^2 \equiv 1 \pmod{15}$.

18. Construct a bijection between the set of even numbers and $\mathbb{N}$. Then, using a diagram, show how integers $\mathbb{Z}$ can be listed in sequence, proving that they are countable.

19. Show that the set of rational numbers $\mathbb{Q}$ is countable by describing an explicit enumeration strategy.

20. Verify that the LuoShu square

4	9	2
3	5	7
8	1	6

    is a magic square. What is its magic sum?

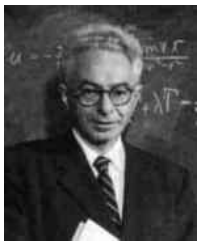
21. Construct a 3×3 magic square modulo 7 using the method of reducing an ordinary magic square. What is the magic sum in $\mathbb{Z}_7$?

22. In a one-time pad over the alphabet $\{A = 0, B = 1, \ldots, Z = 25\}$, encrypt the message `MATH` with the key `CODE`. Show the numerical steps modulo 26 and give the ciphertext.

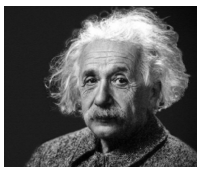
23. Explain why re-using the same key in a one-time pad scheme can destroy perfect secrecy. Give a simple example with short strings.

Chapter 5

We need precision in thought and action to win arguments



"Mathematics as an expression of the human mind reflects the active will, the contemplative reason, and the desire for aesthetic perfection. Its basic elements are logic and intuition, analysis and construction, generality and individuality." – Richard Courant



"Pure mathematics is, in its way, the poetry of logical ideas." – Albert Einstein

Logic is the language of reasoning. Whenever we argue in mathematics, make a claim in science, or even decide something in everyday life, we implicitly use the rules of logic. Boolean logic, named after the mathematician George Boole, provides a precise mathematical framework to study truth and falsity.

In this chapter, we will introduce the basic building blocks of logic, learn how to represent them with truth tables, explore important rules like De Morgan's laws, understand how to work with more than two statements at a time, and examine subtle but important ideas such as vacuous truth. Equally importantly, we will also study some of the standard methods of writing mathematical proofs. These include direct proofs, proofs by counter-examples, proofs by contrapositive, proofs by contradiction, and proofs by induction. Learning these strategies will allow you to apply logical reasoning to establish mathematical results with clarity and rigour.

The aim of this chapter is to provide a foundation that is both rigorous and intuitive, preparing you for deeper study in mathematics and computer science, where careful reasoning and proof techniques are essential.

5.1 Propositions, Basic Boolean Logic and Truth Tables

In mathematics and computer science, many problems boil down to deciding whether something is *true* or *false*. A statement that can be judged true or false is called a *proposition*. For example:

- "2 is an even number" is true.
- "5 is less than 3" is false.
- "$x + 2 = 7$" is a proposition, but its truth depends on the value of x.

Boolean logic studies how we can combine such statements using logical operations like **AND**, **OR**, **NOT**, and **IMPLIES**. These operations are the foundation of digital circuits, programming languages, and formal mathematical proofs.

5.1.1 The basic operations

We denote truth values as T (true) and F (false).

Conjunction (AND). The statement $p \wedge q$ is true only if both p and q are true. *Example:* "I will go for a walk *and* it is sunny." This is only true if both parts are true.

Disjunction (OR). The statement $p \vee q$ is true if at least one of p or q is true. *Example:* "I will have tea *or* coffee." In everyday language, "or" can sometimes mean "one but not both," but in logic the inclusive sense is used: tea, coffee, or both makes the statement true.

Negation (NOT). The statement $\neg p$ has the opposite truth value of p. *Example:* If $p =$"It is raining," then $\neg p =$"It is not raining."

Implication (IF...THEN). The statement $p \Rightarrow q$ means "p implies q" or "If p then q." It is false only when p is true but q is false. In all other cases it is true. Think of it as a promise: "If you pass the QRMT course, then you will get a laddoo."[1] The only way the promise fails is if you pass but still do not get a laddoo. If (unfortunately) you do not pass, the promise is not broken if you still get a laddoo, so the implication is considered true. Passing QRMT is sufficient to get a laddoo, so we say that "p is *sufficient* for q". Passing the course and yet not getting a laddoo will make the promise false, so we say that "q is *necessary* for p".

Biconditional (IF AND ONLY IF). The statement $p \Leftrightarrow q$ is true when both $p \Rightarrow q$ and $q \Rightarrow p$, i.e., p and q have the same truth value, either both true or both false. *Example:* "A number is even if and only if it is divisible by 2."

Exclusive OR (XOR). The statement $p \oplus q$ is true when exactly one of p or q is true, but not both. *Example:* "You can win first prize *or* second prize." You cannot win both at the same time.

5.1.2 Truth tables

Truth tables let us list all possible truth values of statements and see how the logical operations work.

p	q	$p \wedge q$	$p \vee q$	$p \Rightarrow q$	$p \Leftrightarrow q$	$p \oplus q$
T	T	T	T	T	T	F
T	F	F	T	F	F	T
F	T	F	T	T	F	T
F	F	F	F	T	T	F

[1] Note, however, that I am making no such promise. It is just a supposition for making a point.

De Morgan's Laws

These laws explain how negation interacts with AND and OR:

$$\neg(p \wedge q) \equiv (\neg p) \vee (\neg q), \qquad \neg(p \vee q) \equiv (\neg p) \wedge (\neg q).$$

Example 5.1 *Let $p =$ "bring a wizard to the QRMT class" $q =$ "bring a dragon to the QRMT class" Then $\neg(p \vee q)$ means "do not bring a wizard or a dragon to the QRMT class," which is the same as "do not bring a wizard to the QRMT class* and *do not bring a dragon to the QRMT class".*

Example 5.2 *In contrast, $\neg(p \wedge q)$ means "It is not the case that you bring both a wizard and a dragon to the QRMT class," which is logically equivalent to "either you do not bring a wizard to the QRMT class* or *you do not bring a dragon to the QRMT class (or both)."*

Exercise 5.1

1. *Convince yourself that the truth functions given in the table above, and the De Morgan's laws, are reasonable.*
2. *In particular, it is the truth table of $p \Rightarrow q$ that is most commonly used in logical deductions. Try to think of a few example English sentences that corroborate the truth table; especially the third row.*
3. *Argue, using both truth tables and language-based examples, that $p \Rightarrow q$ is equivalent to $\neg q \Rightarrow \neg p$. This is called* contrapositive.
4. *Argue, using both truth tables and language-based examples, that $p \Rightarrow q$ is equivalent to $\neg p \vee q$.*
5. *Argue, using both truth tables and language-based examples, that $p \Leftrightarrow q$ is equivalent to $\neg(p \oplus q)$.*
6. *Sometimes, in everyday life, we observe q, and hypothesize p and $p \Rightarrow q$, i.e., we hypothesize that p may have caused q. This is called* abductive reasoning. *Argue that such abductive reasoning will violate our rules of deduction in mathematics. However, abductive reasoning is essential for the sciences and the social sciences. Can you reason why, perhaps through some examples? The hypotheses generated through abductive reasoning of course need to be validated, even in the sciences and the social sciences.*

The truth tables that underlie propositional logic are not empirical discoveries but conventions inherited from linguistic and philosophical traditions. They represent a form of "agreed upon truth," codifying how we collectively interpret connectives such as "and," "or," "not," and "implies." In this sense they are *axiomatic* in nature: we do not prove that $\neg(p \vee q)$ should behave like $(\neg p \wedge \neg q)$, we simply accept the tabular assignments as the foundation upon which deductions are made. Logical proofs then proceed within this framework of shared agreement. If any one of us refuses to accept these conventions — if, for example, someone insists that "and," "or," or "implies" should work differently — then there is no common ground to move forward, and the very possibility of building logical arguments collapses.

5.1.3 Three-variable examples of truth tables

When we have three propositions p, q, r, there are $2^3 = 8$ possible combinations of truth values. Truth tables become longer, but the method is the same.

Example 1. Consider $p \wedge (q \vee r)$. We first compute $q \vee r$, then combine with p using AND.

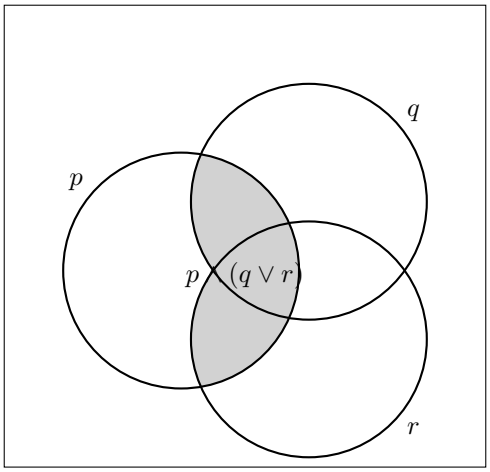
p	q	r	$q \vee r$	$p \wedge (q \vee r)$
T	T	T	T	T
T	T	F	T	T
T	F	T	T	T
T	F	F	F	F
F	T	T	T	F
F	T	F	T	F
F	F	T	T	F
F	F	F	F	F

Example 2. Consider $(p \Rightarrow q) \wedge (q \Rightarrow r)$. This corresponds to chaining two implications: "If p then q, and if q then r."

p	q	r	$p \Rightarrow q$	$q \Rightarrow r$	$(p \Rightarrow q) \wedge (q \Rightarrow r)$
T	T	T	T	T	T
T	T	F	T	F	F
T	F	T	F	T	F
T	F	F	F	T	F
F	T	T	T	T	T
F	T	F	T	F	F
F	F	T	T	T	T
F	F	F	T	T	T

Venn diagram illustration

The formula $p \wedge (q \vee r)$ can also be illustrated with sets. Think of p, q, r as sets of outcomes where each proposition is true. The shaded area corresponds to outcomes where p is true *and* at least one of q or r is true.



5.2 Vacuous Truth

The third line of the truth table for $p \Rightarrow q$, which suggests that if p is *false* and q is *true*, then $p \Rightarrow q$ is *true*, requires special attention. This forms the basis for many subtle logical arguments. A statement of the form "If p then q" is called an implication. As we saw, an implication is only false if p is true and q is false. Therefore, if p is false, the implication is always true. This phenomenon is called *vacuous truth*.

Example 5.3 *1. "If* 5 *is even, then* 5 *is prime." Since* 5 *is not even, the whole statement is true, regardless of the second part.*

2. *"All unicorns have wings." This can be written as "For every x, if x is a unicorn then x has wings." Since there are no unicorns, the condition never applies, and so the statement is vacuously true.*

3. *"If I am the king of France, then $2 + 2 = 5$." Because the condition "I am the king of France" is false, the implication is true.*

Vacuous truth is not a trick — it is a necessary feature of logic. It ensures that statements of the form "All elements of an empty set have property P" are automatically true. For example: "All prime numbers greater than 1000 and less than 1001 are even." There are no such numbers, so the statement is vacuously true.

We will use this quite a bit in the latter chapters.

5.3 Mathematical proofs

A proof is a method for establishing the truth of a statement. We use different methods in different spheres of life:

Rigour	Truth type	Field	Truth teller
0	Word of God	Religion	God/Priests
1	Authoritative truth	Business/School	Boss/Teacher
2	Legal truth	Judiciary	Law/Judge/Lawmakers
3	Philosophical truth	Philosophy	Plausible argument
4	Scientific truth	Physical sciences	Experiments/Observations
5	Statistical truth	Statistics	Data sampling
6	Mathematical truth	Mathematics	Logical deduction

Mathematical reasoning applies the highest standards of rigour for what may be considered as a *proof*, but only for very tightly defined domains. In mathematics we do not accept statements as true merely because they seem plausible or reasonable, or because many examples seem to support them. A *proof* is a logically complete argument that establishes truth from agreed assumptions, which may be definitions, axioms, or previously proved results. This chapter presents some core proof techniques, each with its own "feel" and natural use cases:

1. **Proof by Explicit Construction** — to prove that an element with some qualifying property exists, it is sufficient to construct an example.

2. **Proof by Counter-Example** — to disprove a universal claim, find one instance where it fails.

3. **Direct Proof** — derive the conclusion straight from the hypothesis using definitions and algebra.

4. **Proof by Contradiction** — assume the negation of what you want, reach an impossibility, and conclude the original claim.

5. **Proof by Contrapositive** — prove the logically equivalent statement $\neg Q \Rightarrow \neg P$ instead of $P \Rightarrow Q$.

6. **Proof by the Pigeonhole Principle** — to prove using the principle that if more objects are placed into fewer containers, then at least one container must hold more than one object.

7. **Proof by exhaustion of cases** — prove that statement hold for all of an exhaustive set of cases.

8. **Proof by Induction** — prove a base case and a step that carries truth from n to n+1. We will cover this in the next chapter.

Along the way, we emphasize *how* to think when choosing a technique.

5.3.1 Proof by Explicit Construction

A *proof by explicit construction* demonstrates the truth of a statement of the form "there exists x such that $P(x)$" by actually exhibiting such an x and verifying that it satisfies $P(x)$. This is perhaps the most concrete kind of existence proof: instead of reasoning abstractly, we build or display the required object [2].

In our earlier discussion of *ruler and compass* constructions in Section 3 we showed, step by step, how to construct a length $a+1$ from a given length a, or how to realize the greatest common divisor of two numbers geometrically. Each of those is an instance of explicit construction: the proof of existence of a geometric object lies in the instructions themselves.

Here are some other examples of proof by explicit construction.

Theorem 5.1 *There exist integers x, y such that $14x + 21y = 7$.*

Proof: We explicitly construct one such solution: take $x = -1$, $y = 1$. Then

$$14(-1) + 21(1) = -14 + 21 = 7.$$

Thus such integers exist. In fact, by varying x and y we can generate an infinite family of solutions, but the single explicit example suffices to establish existence. □

Theorem 5.2 *There exist integers a, b, c such that $a^2 + b^2 = c^2$.*

Proof: Exhibit $(a, b, c) = (3, 4, 5)$. Indeed,

$$3^2 + 4^2 = 9 + 16 = 25 = 5^2.$$

Hence such integers exist. This triple is explicitly constructed and is known as the smallest nontrivial Pythagorean triple. □

Theorem 5.3 *For any rationals $r < s$, there exists a rational q with $r < q < s$.*

Proof: Explicitly construct $q = \frac{r+s}{2}$. Because $r < s$, we clearly have $r < q < s$. Moreover, since r and s are rational, their average q is also rational. Thus such a q exists. □

Theorem 5.4 *There exist integers x, y such that*

$$35x + 22y = 1.$$

Proof: To construct a solution, we compute the greatest common divisor of 35 and 22.

$$\begin{aligned} 35 &= 1 \cdot 22 + 13, \\ 22 &= 1 \cdot 13 + 9, \\ 13 &= 1 \cdot 9 + 4, \\ 9 &= 2 \cdot 4 + 1. \end{aligned}$$

Now back-substitute to express 1 as a combination of 35 and 22:

$$1 = 9 - 2 \cdot 4.$$

But $4 = 13 - 1 \cdot 9$, so

$$1 = 9 - 2(13 - 9) = 3 \cdot 9 - 2 \cdot 13.$$

[2] In ancient Mesopotamia (ca 2000 BCE), Babylonian mathematicians used algorithmic procedures to solve linear and quadratic equations for specific examples. The ancient Greeks – most notably Euclid (ca 300 BCE) – introduced geometric construction as a core part of geometric proofs. The kuṭṭaka (pulverizer) algorithm of Āryabhaṭa (5^{th} c) (5^{th} c), Jayadeva (c. 10^{th}–11^{th} c.), and later Bháskara II (12^{th} c.) is a direct construction of integer solutions to linear Diophantine equations. Before the 19^{th} century most proofs were constructive in nature.

Now substitute $9 = 22 - 13$:

$$1 = 3(22 - 13) - 2 \cdot 13 = 3 \cdot 22 - 5 \cdot 13.$$

Next, substitute $13 = 35 - 22$:

$$1 = 3 \cdot 22 - 5(35 - 22) = -5 \cdot 35 + 8 \cdot 22.$$

Thus, one explicit solution is

$$x = -5, \quad y = 8.$$

□

Remark. Equations such as above are called *linear Diophantine equations*. A solution may not always exist. The above method of finding a solution if it exists is called the *Extended Euclidean algorithm*, which not only proves that solutions exist when $\gcd(35, 22) = 1$, but by working through the steps we explicitly *construct* the solution [3].

Theorem 5.5 *The integer* 5 *has a multiplicative inverse modulo* 17.

Proof: We need an integer x such that

$$5x \equiv 1 \pmod{17}.$$

This is equivalent to solving the linear Diophantine equation

$$5x + 17y = 1$$

for integers x, y.

Apply the Euclidean algorithm:

$$17 = 3 \cdot 5 + 2,$$
$$5 = 2 \cdot 2 + 1.$$

Now back-substitute:

$$1 = 5 - 2 \cdot 2.$$

But $2 = 17 - 3 \cdot 5$, so

$$1 = 5 - 2(17 - 3 \cdot 5) = 7 \cdot 5 - 2 \cdot 17.$$

Thus

$$1 = 7 \cdot 5 + (-2) \cdot 17,$$

which shows that $x = 7$, $y = -2$ is an explicit solution.

Therefore

$$5 \cdot 7 \equiv 1 \pmod{17},$$

so 7 is the multiplicative inverse of 5 modulo 17. □

Remark. This construction not only proves that 5 has an inverse modulo 17, but also produces the explicit inverse 7. In fact, the extended Euclidean algorithm always gives such a construction whenever $\gcd(a, m) = 1$.

[3] The origins of this algorithm can be traced back to the **kuṭṭaka** (pulverizer) algorithm of Āryabhaṭa (5^{th} c), Jayadeva (c. 10^{th}–11^{th} c.) and later Bháskara II (12^{th} c.) refined and applied the kuṭṭaka extensively. Bháskara in the Līlāvatī and Bījagaṇita gives worked examples of solving linear Diophantine equations with what is recognizably the extended Euclid method.

When to prove by explicit construction. Proof by explicit construction is particularly valuable when the question is "does there exist?" and the object in question is concrete enough to build or write down. It contrasts with nonconstructive methods (such as contradiction or pigeonhole arguments, which we will study later) where existence is established without showing a specific example. Both are mathematically valid, but constructive proofs have the added advantage of providing insight into the nature of the object itself.

5.3.2 Proof by Counter-Example

A universal statement $\forall x\, P(x)$ is false if there exists a single x with $\neg P(x)$. Producing such an x *disproves* the claim completely [4]. For example, consider the exchange of Figure 5.1 on social media

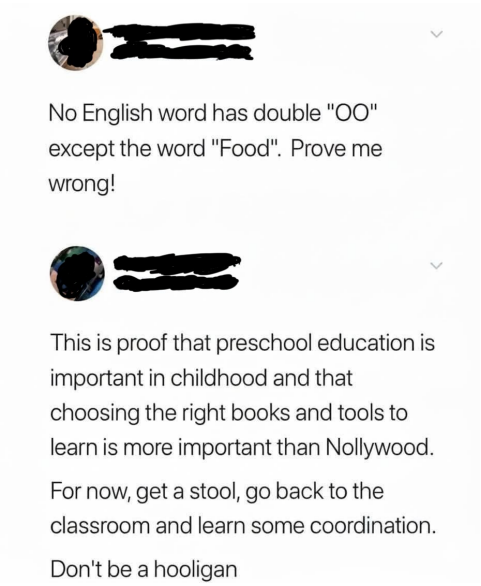


Figure 5.1: A counter-example to a universal claim

Exercise 5.2 *Which basic logical reasoning method has been used here to disprove the universal claim?*

Example 5.4 *Let us consider a few more examples.*

1. **Claim***: "The product of any two prime numbers is prime."*
   Counter-example. $2 \cdot 3 = 6$ *is not prime.*

2. **Claim***: "For all integers n, the number $n^2 + n + 41$ is prime."*
   Counter-example. *At $n = 41$, $41^2 + 41 + 41 = 41 \cdot 43$ is composite.*

.

What counter-examples teach. They refine sloppy universal claims. When a statement fails, analyzing *why* the witness breaks it often suggests a corrected statement. The expression $n^2 + n + 41$ produces prime numbers for all integers n with $0 \leq n \leq 39$, but fails afterwards.

[4] Philosophically and logically, the concept of a counterexample existed in ancient Greek dialectics and argumentation, notably in the Socratic method where contradictory examples were used to challenge general claims or definitions. In medieval and early modern mathematics, counterexamples became more systematically used to falsify conjectures. Famous historical examples include Euler's counterexample disproving the conjecture that all Fermat numbers are prime.

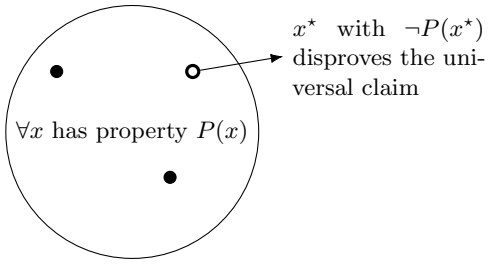


Figure 5.2: One counterexample $x^\star$ inside the "universe" circle shows the claim $\forall x\, P(x)$ is false.

5.3.3 Direct Proof

A direct proof of $P \Rightarrow Q$ proceeds as a straight line: Assume P $\rightarrow$ unpack definitions $\rightarrow$ algebra/logic $\rightarrow Q$.

Consider the following examples.

Definition. An integer n is *even* if $n = 2k$ for some integer k; it is *odd* if $n = 2k + 1$ for some integer k.

Theorem 5.6 *If a and b are even integers, then $a + b$ is even.*

Proof: Let $a = 2k$ and $b = 2m$ for integers k, m. Then

$$a + b = 2k + 2m = 2(k + m),$$

which is a multiple of 2, hence even. □

Theorem 5.7 *If x is odd, then x^2 is odd.*

Proof: Let $x = 2k + 1$. Then $x^2 = (2k + 1)^2 = 4k^2 + 4k + 1 = 2(2k^2 + 2k) + 1$, which is odd. □

Theorem 5.8 $1 + 2 + 3 + \cdots + n = n(n + 1)/2$

Proof: Let $S = 1 + 2 + 3 + \cdots + n$.
Hence, $S = n + (n - 1) + (n - 2) + \cdots + 1$, in the reverse order. This implies that

$$2S = \underbrace{(n + 1) + (n + 1) + (n + 1) + \cdots + (n + 1)}_{n \text{ times}}$$

Thus, $S = n(n + 1)/2$. □

Theorem 5.9 *Every odd integer is equal to the difference between the squares of two integers.*

Proof: Let $a = 2k+1$ be an arbitrary odd integer. Then $a = 2k+1 = k^2+2k+1-k^2 = (k+1)^2-k^2$.
□

When to prefer direct proofs. They shine when definitions already carry the structure you need, or when simple algebra lifts P to Q. If you find yourself repeatedly "expanding the definitions" as your first move, you are in direct-proof territory.

5.3.4 Proof by Contradiction

To prove $P \Rightarrow Q$ by contradiction, assume both P and $\neg Q$. If these assumptions force an impossibility (a statement that cannot be true), then $\neg Q$ must be false, hence Q true [5]. This style of reasoning is also called reductio ad absurdum.

Consider the following examples.

Theorem 5.10 *If n is odd, then n^2 is odd.*

Proof: Let P be the statement "n is odd," and Q be the statement "n^2 is odd." Assume $P \wedge \neg Q$: that is, n is odd but n^2 is even. If n is odd, then $n = 2k + 1$ for some integer k. Then

$$n^2 = (2k+1)^2 = 4k^2 + 4k + 1 = 2(2k^2 + 2k) + 1,$$

which is odd. This contradicts the assumption that n^2 is even. Therefore, n^2 must be odd. □

Theorem 5.11 *If ab is odd, then both a and b are odd.*

Proof: Let P be the statement "ab is odd," and Q be the statement "both a and b are odd." Assume $P \wedge \neg Q$: that is, ab is odd but at least one of a or b is even. If a is even, say $a = 2k$, then $ab = 2kb$ is even, contradiction. Similarly, if b is even then ab is even, contradiction. Therefore, both a and b must be odd. □

Theorem 5.12 *If n^2 is divisible by 3, then n is divisible by 3.*

Proof: Let P be the statement "n^2 is divisible by 3," and Q be the statement "n is divisible by 3." Assume $P \wedge \neg Q$: that is, n^2 is divisible by 3 but n is not divisible by 3. If n is not divisible by 3, then $n \equiv 1 \pmod 3$ or $n \equiv 2 \pmod 3$. In both cases,

$$n^2 \equiv 1 \pmod 3,$$

so n^2 is not divisible by 3. This contradicts our assumption. Therefore, n must be divisible by 3. □

Often, the antecedent P need not be explicit in contradiction proofs. In what follows, we prove that $\sqrt{2}$ which we constructed using ruler and compass in Exercise 3.5 is not a rational number.

Theorem 5.13 *$\sqrt{2}$ is irrational.*

Proof: Assume $\sqrt{2} = \frac{p}{q}$ in lowest terms with $p, q \in \mathbb{Z}_{>0}, gcd(p, q) = 1$. Then $p^2 = 2q^2$, so p is even; write $p = 2r$. Substituting, $4r^2 = 2q^2$ gives $q^2 = 2r^2$, hence q is even. Thus p, q share a factor 2, contradicting lowest terms. □

In this example the antecedent P is the silent statement "the number $\sqrt{2}$ exists", and Q is the statement that $\sqrt{2}$ is irrational.

Also consider the following examples:

Theorem 5.14 *Every integer $n > 1$ has at least one prime factor.*

Proof: Let $n > 1$. If n is prime, then it is its own prime factor.

Otherwise, suppose n is composite. Let d be the smallest divisor of n greater than 1. By definition, d divides n. We claim that d must be prime.

Suppose, for contradiction, that d is composite. Then $d = ab$ with $1 < a < d$ and $1 < b < d$. But then a divides d, and since d divides n, we also have a divides n. This contradicts the minimality of d, because a is a smaller divisor of n greater than 1.

Therefore d must be prime, and hence n has a prime factor in all cases. □

[5] Euclid's Elements contains many early examples of proof by contradiction. For instance, in Book 1, Proposition 6, Euclid proves that if two angles of a triangle are equal, the sides opposite these angles are equal by assuming the contrary and deriving a contradiction. In the section we illustrate two other famous one's – *$\sqrt{2}$ is irrational* and *there are infinitely many primes*. In Indian traditions too, the Jaina mathematicians (c. 6th–9th c.) often used impossibility reasoning, e.g. showing that certain infinite processes yield contradictions, to motivate definitions of infinity and infinitesimals. Bháskara's arguments about the impossibility of certain rational approximations can also be read as of the reductio-style.

Theorem 5.15 *Let p be a prime. If p divides n, then p does not divide $n+1$.*

Proof: Suppose p divides n and p divides $n+1$. Then, for some integers a and b, $n = pa$ and $n+1 = pb$.

We then have that $(n+1) - n = 1 = p(b-a)$, or p divides 1, which is impossible. □

Theorem 5.16 *There exist infinitely many prime numbers.*

Proof:

Assume, to the contrary, that there are only finitely many primes $p_1, p_2, \ldots, p_n$. Consider the number

$$N = p_1 p_2 \cdots p_n + 1.$$

Clearly $N > 1$ and hence must have a prime divisor. But no p_i divides N, since dividing N by any p_i leaves a remainder 1. This contradicts the assumption that $p_1, \ldots, p_n$ were all the primes. Therefore, there must exist infinitely many primes [6] □

Theorem 5.17 *There exists an irrational number x such that x^2 is rational.*

Proof: Suppose, for contradiction, that no such number exists; that is, whenever x is irrational, x^2 is irrational as well. Consider $x = \sqrt{2}$. Then x is irrational, but

$$x^2 = (\sqrt{2})^2 = 2,$$

which is rational. This contradicts our assumption. Hence, there does exist an irrational number x such that x^2 is rational. □

Exercise 5.3 *What are the antecedents P and consequents Q in Theorems 5.16 and 5.17.*

When to use contradiction. Contradiction is powerful for showing that something is impossible – for example irrationality, no smallest positive rational, etc. – but it also works beautifully for proving existence: to show "there exists an object with property P," assume that no such object exists, and derive an inconsistency.

5.3.5 Proof by Contrapositive

In Exercise 5.1 we argued that implications $P \Rightarrow Q$ and $\neg Q \Rightarrow \neg P$ are logically equivalent. Sometimes $\neg Q \Rightarrow \neg P$ is cleaner because "failing Q" has a rigid structure (e.g., divisibility, parity), while Q itself would require awkward casework.

Theorem 5.18 *If n^2 is odd, then n is odd.*

Proof: Assume that n is even, i.e., $n = 2k$ for some k. Then, $n^2 = 4k^2 = 2(2k^2)$ is also even. Hence the contrapositive $\neg Q \Rightarrow \neg P$ holds, and thus the original implication. □

Theorem 5.19 *If n^2 is divisible by 3, then n is divisible by 3.*

Proof: Assume that n is not divisible by 3. Then $n \equiv 1$ or $2 \pmod 3$. In both cases $n^2 \equiv 1 \pmod 3$, so n^2 is not divisible by 3. Hence the contrapositive $\neg Q \Rightarrow \neg P$ holds, and thus the original implication. □

Theorem 5.20 *If n does not divide ab, then n does not divide a and n does not divide b.*

[6] This proof was given by Euclid. This is also the first proof from the Proofs from the Book, which was written in the memory of the Hungarian mathematician Paul Erdős, who liked to talk about The Book, in which God maintains the perfect proofs for mathematical theorems.

Proof: Let (n divides a) or (n divides b).

If n divides a, then $a = nd$ for some $d > 0$. Thus $ab = ndb = n(db)$, which implies that n divides ab.

If n divides b, argue similarly.

□

Theorem 5.21 *Let $n \in \mathbb{Z}$. If $n^2 - 6n + 5$ is even, then n is odd.*

Proof: Let n be even, i.e., $n = 2k$ for some k. Then, $n^2 - 6n + 5 = (2k)^2 + 6(2k) + 4 + 1 = 2(2k^2 + 6k + 2) + 1$ which is odd. Hence the contrapositive is proved. □

Exercise 5.4 *Argue that whatever can be proved by* contradiction *can also be proved by* contrapositive*, and vice versa.*

Choosing contrapositive. Look for conclusions phrased as "is divisible by ·, is even, is nonnegative, is a subset of ·"—their negations often have crisp arithmetical or set-theoretic descriptions that are easy to exploit.

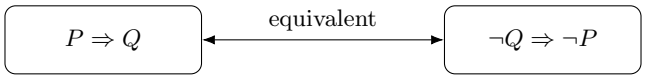


Figure 5.3: Proving the contrapositive proves the original implication.

5.3.6 Proof by the Pigeonhole Principle

The *Pigeonhole Principle* states that if $n + 1$ or more objects are placed into n boxes, then some box must contain at least two objects. More generally, if N objects are placed into k boxes, then some box contains at least $\lceil N/k \rceil$ objects[7]. The pigeonhole principle itself can be proved by either contradiction or contrapositive [8].

Exercise 5.5 *Prove the pigeonhole principle using both contradiction and contrapositive.*

The pigeonhole principle however merits an independent consideration because this special form makes several proofs easier. This principle often provides quick existence proofs when explicit construction is difficult. It guarantees that a certain configuration must exist, even if we cannot point to the exact example.

Example 5.5 *In a group of 13 people, at least two must share a birth month. Here the "boxes" are the 12 months, and the "objects" are the 13 people. By the pigeonhole principle, some month contains at least two birthdays.*

Here are a few more examples.

Theorem 5.22 *Among any $n + 1$ integers, there exist two with the same remainder when divided by n.*

Proof: The possible remainders upon division by n are $0, 1, 2, \ldots, n - 1$, giving n "boxes." Placing $n + 1$ integers into these boxes, two must land in the same box. Therefore two of the integers have the same remainder. □

[7] $\lceil N/k \rceil$ is the smallest integer q such that $qk \geq N$.

[8] The first known formal appearance of the pigeonhole is often attributed to the German mathematician Peter Gustav Lejeune Dirichlet in 1834, who called it the "Schubfachprinzip" (drawer or box principle). The principle appears indirectly and informally much earlier, dating back at least to 1622 in a Latin work by the French Jesuit mathematician Jean Leurechon. But Indian combinatorial work (e.g. Pingala's prosody, c. 200 BCE) already involved reasoning about distributing syllables into patterns — essentially counting arrangements that could be seen pigeonhole-wise. Later, in combinatorial discussions in the Chandas-śāstra and in Bháskara's combinatorics, one finds implicit arguments about "if you have more patterns than slots, something repeats."

Theorem 5.23 *Given n integers $a_1, a_2, \ldots, a_n$, there exists a non-empty subset whose sum is divisible by n.*

Proof: Consider the partial sums

$$s_k = a_1 + a_2 + \cdots + a_k, \quad k = 1, 2, \ldots, n.$$

There are n such sums. If any s_k is divisible by n, we are done. Otherwise, each s_k has a remainder in $\{1, 2, \ldots, n-1\}$. By the pigeonhole principle, two of the s_i, s_j (say $i < j$) have the same remainder. Then $s_j - s_i$ is divisible by n, and this difference is the sum of the subset $\{a_{i+1}, \ldots, a_j\}$. Hence such a subset always exists. □

Theorem 5.24 *In any group of n people, if friendship is always mutual, then at least two people have the same number of friends.*

Proof: Each person can have between 0 and $n-1$ friends. But it is impossible to have simultaneously one person with 0 friends and another with $n-1$ friends, since the "friend of all" would have to be friends with the "friend of none." Therefore, the possible friend counts are at most $n-1$ distinct numbers. With n people, two must share the same friend count. □

5.3.7 Proof by Exhaustion of cases

Sometimes a proposition cannot be proved in one uniform argument, but instead requires us to split the possible situations into a finite number of cases. A *proof by exhaustion of cases* works by checking each case separately and showing that the desired conclusion holds in all of them [9]. The structure is:

1. Partition the domain of the problem into finitely many exhaustive and mutually exclusive cases.
2. Prove the statement in each case individually using any of the proof techniques.
3. Conclude that the statement holds in general.

This method is indispensable when a property depends on a small number of discrete possibilities, such as parity (even/odd), sign (positive/negative/zero), or congruence classes modulo n.

Let us consider a few examples.

Theorem 5.25 *For any integer n, the number $n^2 + n$ is even.*

Proof: We consider two exhaustive cases:

- **Case 1:** n is even. Then $n = 2k$ for some integer k. So $n^2 + n = (2k)^2 + 2k = 4k^2 + 2k = 2(2k^2 + k)$, which is even.
- **Case 2:** n is odd. Then $n = 2k + 1$ for some integer k. So $n^2 + n = (2k+1)^2 + (2k+1) = 4k^2 + 6k + 2 = 2(2k^2 + 3k + 1)$, which is even.

In either case, $n^2 + n$ is even. Therefore the theorem holds. □

Theorem 5.26 *For any integers a and b, $|a| \cdot |b| = |ab|$.*

Proof: We proceed by cases on the signs of a and b:

[9] The earliest history of the method of exhaustion can be traced back to Antiphon of Athens (ca. 480-411 BCE), but the full logical method as a form of a rigorous proof was formalised by Eudoxus of Cnidus (ca 408-355 BCE). There are several examples in Euclid's Elements and the work of Archimedes. Āryabhaṭa (5th c.) and Bháskara II (12th c.) used case-based reasoning to establish divisibility properties and to reduce large problems to smaller congruence classes. In the Bījagaṇita, Bháskara explicitly gives different rules depending on whether numbers are odd or even – an early form of modular case-splitting. The most notable example of proof by case analysis is the modern computer aided proof of the four colour theorem.

- **Case 1:** $a \geq 0$, $b \geq 0$. Then $|a| = a$, $|b| = b$, so $|a||b| = ab = |ab|$.
- **Case 2:** $a \geq 0$, $b < 0$. Then $|a| = a$, $|b| = -b$, and $|a||b| = a(-b) = -ab$. Since $ab < 0$, $|ab| = -ab$, so equality holds.
- **Case 3:** $a < 0$, $b \geq 0$. Then $|a| = -a$, $|b| = b$, and $|a||b| = (-a)b = -ab$. Again $ab < 0$, so $|ab| = -ab$, equality holds.
- **Case 4:** $a < 0$, $b < 0$. Then $|a| = -a$, $|b| = -b$, so $|a||b| = (-a)(-b) = ab$. Here $ab > 0$, so $|ab| = ab$, equality holds.

In all possible cases, $|a||b| = |ab|$. □

Theorem 5.27 *There is no solution in integers to* $(x^2 - y^2)$ *mod* $4 = 2$.

Proof:

- **Case 1:** x is even and y is even $\Rightarrow x^2 = 4m, y^2 = 4n$ for some integers m and $n \Rightarrow (x^2 - y^2) = 4(m - n)$.
- **Case 2:** x is even and y is odd $\Rightarrow x^2 = 4m, y^2 = 4n + 1$ for some integers m and $n \Rightarrow (x^2 - y^2) = 4(m - n) - 1$.
- **Case 3:** x is odd and y is even $\Rightarrow x^2 = 4m + 1, y^2 = 4n$ for some integers m and $n \Rightarrow (x^2 - y^2) = 4(m - n) + 1$.
- **Case 4:** x is odd and y is odd $\Rightarrow x^2 = 4m + 1, y^2 = 4n + 1$ for some integers m and $n \Rightarrow (x^2 - y^2) = 4(m - n)$.

In all these four cases $(x^2 - y^2) \bmod 4 \neq 2$. □

Theorem 5.28 *An irrational raised to an irrational power may be rational.*

Proof: We already know that $\sqrt{2}$ is irrational. Let $a = \sqrt{2}^{\sqrt{2}}$. Two cases arise.

- **Case 1:** a is rational. Then, the proposition is clearly true.
- **Case 2:** a is irrational. In that case consider $a^{\sqrt{2}} = (\sqrt{2}^{\sqrt{2}})^{\sqrt{2}} = \sqrt{2}^{\sqrt{2}\sqrt{2}} = \sqrt{2}^2 = 2$ which is a rational.

□

When to use case analysis. Proof by exhaustion is most effective when the number of cases is small and natural, such as two parities, three sign possibilities, or finitely many congruence classes modulo n. However, it becomes impractical if the number of cases grows large. As a general strategy, one should strive to minimize the cases by exploiting symmetry, structure, or general arguments.

5.3.8 Proofs of Equivalence

Often in mathematics we want to prove a statement of the form

$$P \iff Q,$$

which reads "P if and only if Q" (abbreviated "P iff Q"). This means both $P \Rightarrow Q$ and $Q \Rightarrow P$ hold. To prove such an equivalence, we usually split the work into two parts:

1. Prove the *forward implication* $P \Rightarrow Q$.
2. Prove the *reverse implication* $Q \Rightarrow P$.

Equivalences require a special discussion because sometimes the two directions may use very different arguments. Consider the following examples.

Theorem 5.29 *An integer n is even $\iff$ n^2 is even.*

Proof: ($\Rightarrow$) Suppose n is even, say $n = 2k$. Then $n^2 = (2k)^2 = 4k^2 = 2(2k^2)$ is even.

($\Leftarrow$) Conversely, suppose n^2 is even. If n were odd, say $n = 2k + 1$, then $n^2 = (2k + 1)^2 = 4k^2 + 4k + 1 = 2(2k^2 + 2k) + 1$, which is odd — a contradiction. Hence n must be even. □

Theorem 5.30 *An integer n is divisible by 3 $\iff$ the sum of its digits is divisible by 3.*

Proof: Write n in decimal expansion as

$$n = d_0 + 10d_1 + 10^2 d_2 + \cdots + 10^k d_k,$$

where $d_0, d_1, \ldots, d_k$ are the digits of n.

Since $10 \equiv 1 \pmod 3$, it follows that

$$10^i \equiv 1 \pmod 3 \quad \text{for every } i \geq 0.$$

Therefore

$$n \equiv d_0 + d_1 + \cdots + d_k \pmod 3.$$

($\Rightarrow$) Suppose n is divisible by 3. Then $n \equiv 0 \pmod 3$. But $n \equiv d_0 + d_1 + \cdots + d_k \pmod 3$. Hence the digit sum is also congruent to 0 modulo 3, i.e. divisible by 3.

($\Leftarrow$) Conversely, suppose the digit sum is divisible by 3. Then $d_0 + d_1 + \cdots + d_k \equiv 0 \pmod 3$. Since $n \equiv d_0 + \cdots + d_k \pmod 3$, it follows that $n \equiv 0 \pmod 3$, i.e. n is divisible by 3. □

Theorem 5.31 *Let n be a positive integer with decimal form $n = 10q + u$, where u is the units digit and q is the number formed by the remaining digits. Then*

$$n \text{ is divisible by } 7 \iff q - 2u \text{ is divisible by } 7.$$

Proof: Working modulo 7, note first that $10 \equiv 3 \pmod 7$, hence

$$n = 10q + u \equiv 3q + u \pmod 7.$$

Compute the difference

$$(3q + u) - (q - 2u) = 2q + 3u = 2(q - 2u) + 7u.$$

Therefore

$$3q + u \equiv 2(q - 2u) \pmod 7.$$

Since 2 is invertible modulo 7 (indeed $2^{-1} \equiv 4 \pmod 7$), we have

$$3q + u \equiv 0 \pmod 7 \iff q - 2u \equiv 0 \pmod 7.$$

Combining with $n \equiv 3q + u \pmod 7$ yields

$$n \equiv 0 \pmod 7 \iff q - 2u \equiv 0 \pmod 7,$$

Then, ($\Rightarrow$) If n is divisible by 7, then $n \equiv 0 \pmod 7$. Hence $q - 2u \equiv 0 \pmod 7$, i.e., $q - 2u$ is divisible by 7.

($\Leftarrow$) Conversely, if $q - 2u$ is divisible by 7, then $q - 2u \equiv 0 \pmod 7$. Since $n \equiv q - 2u \pmod 7$, it follows that $n \equiv 0 \pmod 7$, i.e., n is divisible by 7.

□

Example. For $n = 483$ we have $q = 48$, $u = 3$, so $q - 2u = 48 - 6 = 42$, a multiple of 7. Hence 483 is divisible by 7.

Exercise 5.6 *Argue that for bigger numbers, the test can be applied repeatedly by computing $q - 2u$ on the result.*

Discussion. When faced with an equivalence, it is often useful to remember that "$P \iff Q$" is logically the same as "$(P \Rightarrow Q) \wedge (Q \Rightarrow P)$." It is also common to use *contrapositive arguments* in one or both directions.

5.3.9 Proof by Induction

Proof by induction is a powerful technique for proving statements that are asserted to be true for all integers $n \geq n_0$. The method rests on two steps: establishing a base case and showing that if the statement holds for an arbitrary integer k, then it must also hold for $k + 1$.

The Induction Principle

Suppose $P(n)$ is a statement depending on an integer n. To prove that $P(n)$ is true for all integers $n \geq n_0$, we proceed as follows:

1. **Base case:** Verify that $P(n_0)$ is true.
2. **Induction step:** Assume $P(k)$ is true for some arbitrary integer $k \geq n_0$ (this assumption is called the *induction hypothesis*), and then show that $P(k + 1)$ must also be true.

If both steps succeed, then by the principle of mathematical induction, $P(n)$ is true for all integers $n \geq n_0$ [10]. Think of a staircase: show you can step onto the first stair (base case) and that from any stair you can step to the next (inductive step). Then you can reach all stairs.

Exercise 5.7 *Prove that strong induction is equivalent to the original induction principle.*

Let us consider a few examples of proof by induction below.

Theorem 5.32 *For all $n \geq 1$,*

$$1 + 2 + \cdots + n = \frac{n(n+1)}{2}.$$

Proof:
Base case: For $n = 1$, the left-hand side is 1, and the right-hand side is $\frac{1 \cdot 2}{2} = 1$. So the statement holds.
Induction hypothesis: Assume the formula holds for $n = k$, i.e.

$$1 + 2 + \cdots + k = \frac{k(k+1)}{2}.$$

Induction step: Now consider $n = k + 1$:

$$1 + 2 + \cdots + k + (k+1) = \frac{k(k+1)}{2} + (k+1) = \frac{k(k+1) + 2(k+1)}{2} = \frac{(k+1)(k+2)}{2}.$$

This is exactly the formula with $n = k + 1$. Hence the result follows. □

[10] The idea of proof by induction is ancient, though the formal principle was only stated in the nineteenth century. In the Western tradition, inductive reasoning appears already in Euclid's *Elements* (Book IX, Proposition 8) and in his descent-style proof of prime factorisation (Book VII, Propositions 30–32). Later, Islamic mathematicians such as al-Karaji (10th–11th c.) used inductive arguments for binomial expansions, and in Europe, Maurolico (16th c.) and Pascal (17th c.) applied the method in combinatorics and series. By the time of Euler and Gauss, induction was common in number theory, and Peano (1889) finally codified it as an axiom of the natural numbers. In the Indian tradition, recursive and inductive reasoning also played a central role: Pingala (c. 200 BCE) described prosodic patterns equivalent to Pascal's triangle, and Bhāskara II (12th c.) used stepwise arguments in the *Līlāvatī* and *Bījagaṇita* to establish general formulas. Thus, both traditions employed inductive reasoning long before its modern formal axiomatization.

Theorem 5.33 *For all $n \geq 1$,*

$$1 + 3 + 5 + \cdots + (2n - 1) = n^2.$$

Proof:
Base case: For $n = 1$, the left-hand side is 1 and the right-hand side is $1^2 = 1$.
Induction hypothesis: Assume true for $n = k$, i.e.

$$1 + 3 + \cdots + (2k - 1) = k^2.$$

Induction step: Then for $n = k + 1$,

$$1 + 3 + \cdots + (2k - 1) + (2(k + 1) - 1) = k^2 + (2k + 1) = (k + 1)^2.$$

Thus the claim holds for $n = k + 1$. □

Theorem 5.34 *For all $n \geq 1$, $10^n - 1$ is divisible by 9.*

Proof:
Base case: For $n = 1$, $10^1 - 1 = 9$, which is divisible by 9.
Induction hypothesis: Assume $10^k - 1$ is divisible by 9, i.e., $10^k - 1 = 9m$ for some m.
Induction step: Then

$$10^{k+1} - 1 = 10 \cdot 10^k - 1 = 10(9m + 1) - 1 = 90m + 9 = 9(10m + 1).$$

Thus $10^{k+1} - 1$ is divisible by 9. □

Theorem 5.35 *For all integers $n \geq 4$, $2^n > n^2$.*

Proof:
Base case: For $n = 4$, $2^4 = 16$ and $4^2 = 16$, so the inequality holds with equality. For $n = 5$, $2^5 = 32 > 25$, so the inequality holds strictly.
Induction hypothesis: Assume $2^k > k^2$ for some $k \geq 5$.
Induction step: Then

$$2^{k+1} = 2 \cdot 2^k > 2 \cdot k^2.$$

Since $k \geq 5$, we have $2k^2 \geq (k + 1)^2$. Thus $2^{k+1} > (k + 1)^2$. □

Theorem 5.36 *Suppose we have stamps of two different denominations, 3 paise and 5 paise. We want to show that it is possible to make up exactly any postage of 8 paise or more using stamps of these two denominations. Thus we want to show that every positive integer $n \geq 8$ is expressible as $n = 3i + 5j$ where $i, j \geq 0$.*

Proof:
Base case: For $n = 8$, we have $n = 3 + 5$, i.e. $i = j = 1$.
Induction hypothesis: $n = 3i + 5j$ for an $n \geq 8$, $i, j \geq 0$.
Induction step: Consider $n+1$. If $j = 0$ then clearly $i \geq 3$ and we may write $n+1$ as $3(i-3)+5(j+2)$. Otherwise $n + 1 = 3(i + 2) + 5(j - 1)$. □

Strong induction. Strong induction is a variant where we assume the statement holds for all integers up to k and use this to prove it for $k + 1$. Suppose $P(n)$ is a statement depending on an integer n. To prove that $P(n)$ is true for all integers $n \geq n_0$, we proceed as follows:

1. **Base case:** Verify that $P(n_0)$ is true.
2. **Induction step:** Assume $P(m)$ is true for all m, $n_0 \leq m \leq k$ for some arbitrary integer k (this assumption is called the strong version of the *induction hypothesis*), and then show that $P(k + 1)$ must also be true.

Consider the following example.

Theorem 5.37 *Let $F_0 = 0,\ F_1 = 1,\ F_2 = 1, \ldots$ be the Fibonacci sequence where for all $n \geq 2$, $F_n = F_{n-1} + F_{n-2}$. Let $\phi = (1+\sqrt{5})/2$. We now show that $F_n \leq \phi^{n-1}$ for all positive n.*

Proof:
Base case: For $n = 1$, we have $F_1 = \phi^0 = 1$.
Induction hypothesis: $F_n \leq \phi^{n-1}$ for all m, $1 \leq m \leq n$.
Induction step:

$$\begin{aligned} F_{n+1} &= F_n + F_{n-1} \\ &\leq \phi^{n-1} + \phi^{n-2} && \text{(by the induction hypothesis)} \\ &= \phi^{n-2}(\phi + 1) \\ &= \phi^n && \text{(since } \phi^2 = \phi + 1\text{)} \end{aligned}$$

□

Theorem 5.38 (Fundamental Theorem of Arithmetic) *Every integer $n > 1$ can be written as a product of primes, and this representation is unique up to reordering of the primes.*

Proof: *Existence:* We proceed by induction on n. For $n = 2$, the result holds since 2 is prime. Assume every integer m with $2 \leq m < n$ has a prime factorisation. If n is prime, we are done. Otherwise, $n = ab$ with $1 < a, b < n$. By the induction hypothesis, both a and b factor into primes. Thus n factors into primes.

Uniqueness: Suppose

$$n = p_1 p_2 \cdots p_r = q_1 q_2 \cdots q_s,$$

with all p_i, q_j prime. Since $p_1 \mid q_1 q_2 \cdots q_s$, Euclid's Lemma implies $p_1 \mid q_j$ for some j. As both are prime, $p_1 = q_j$. Cancelling p_1 from both sides and repeating the argument gives uniqueness by induction on the number of prime factors. □

Exercise 5.8 *Complete the proof of the unique prime factorization theorem more formally.*

Remarks.

- Induction proofs are especially useful for formulas involving sums, products, inequalities, or divisibility.
- The induction hypothesis is a temporary assumption used to prove the next case; it is not assumed globally.
- Strong induction is a variant where we assume the statement holds for all integers up to k and use this to prove it for $k + 1$.

5.3.10 Conclusion

There is no single "best" technique; experienced problem solvers try viewpoints. If definitions seem aligned, push a direct proof. If "not Q" feels structured, flip to a contrapositive. If a statement ranges over integers, try induction. If you suspect a claim is too strong, hunt a counter-example. And whenever "assuming the opposite" quickly generates an impossibility, contradiction is often the sharpest tool.

Problems

1. **Truth Tables**
   (a) Construct the truth table for $(p \wedge q) \vee (\neg r)$.
   (b) Verify by a truth table that $(p \Rightarrow r) \wedge (q \Rightarrow r)$ is equivalent to $(p \vee q) \Rightarrow r$.

(c) Determine whether $(p \Rightarrow q) \Rightarrow r$ and $p \Rightarrow (q \Rightarrow r)$ are equivalent.

(d) Show by truth table that $p \vee (q \wedge r)$ is not equivalent to $(p \vee q) \wedge (p \vee r)$.

2. **Three-variable Logic**

(a) Construct the truth table for $(p \oplus q) \oplus r$ and check whether it is associative.

(b) Verify that $\neg(p \wedge q \wedge r)$ is equivalent to $\neg p \vee \neg q \vee \neg r$.

(c) Show that $(p \Rightarrow q) \wedge (q \Rightarrow r)$ does not imply $(p \Rightarrow r)$ by giving a truth table.

(d) Identify all assignments of p, q, r for which $(p \vee q) \Rightarrow (q \vee r)$ is false.

3. **Vacuous Truth**

(a) State a universally quantified claim about an empty set and explain why it is true.

(b) Explain why the statement "If n is an integer with $n^2 = -1$, then n is prime" is vacuously true.

(c) Let $A = \varnothing$. Prove that "For all $x \in A$, $x^2 = 0$" is vacuously true.

(d) Formulate a vacuous truth involving divisibility (e.g. "All integers divisible by both 2 and 3 and equal to 5 are even"), and justify why it is vacuously true.

4. **Proof by Explicit Construction**

(a) Find integers x, y such that $17x + 29y = 1$.

(b) Exhibit an explicit Pythagorean triple different from $(3, 4, 5)$ and $(5, 12, 13)$.

(c) Construct a rational number between $\frac{7}{9}$ and $\frac{8}{9}$.

(d) Show by construction that there exists an integer solution to $12x + 18y = 6$.

5. **Proof by Counter-Example**

(a) Disprove: "For all integers n, $n^2 - n + 41$ is prime."

(b) Disprove: "For all integers n, $n^3 + 2$ is prime."

(c) Disprove: "Every integer greater than 2 is the sum of two primes." (Give a counterexample.)

(d) Find a counterexample to: "If a divides bc, then a divides b."

6. **Direct Proof**

(a) Prove that the product of two even integers is even.

(b) Prove that if a is divisible by 12 and b is divisible by 6, then $a + b$ is divisible by 6.

(c) Show directly that if n is a multiple of 4, then n^2 is a multiple of 16.

(d) Prove directly that if $a \equiv b \pmod{m}$ and $c \equiv d \pmod{m}$, then $a + c \equiv b + d \pmod{m}$.

7. **Proof by Contradiction**

(a) Prove by contradiction that $\sqrt{7}$ is irrational.

(b) Show by contradiction that there is no integer solution to $2x = 2y + 1$.

(c) Prove by contradiction that there are infinitely many primes congruent to 1 modulo 4.

(d) Prove by contradiction that $\sqrt[3]{2}$ is irrational.

8. **Proof by Contrapositive**

(a) Prove: If n^2 is divisible by 9, then n is divisible by 3.

(b) Prove: If ab is even, then at least one of a or b is even.

(c) Prove: If n^2 is divisible by 8, then n is divisible by 2.

(d) Prove: If x^2 is divisible by 12, then x is divisible by 6.

9. **Proof by the Pigeonhole Principle**

   (a) Show that among any 8 integers, two leave the same remainder modulo 7.

   (b) Prove that in a group of 32 people, at least two share the same day of the month for their birthday.

   (c) Show that in any set of 6 integers, at least three must have the same parity.

   (d) Prove that in any group of 367 people, at least two have the same exact birthday (ignoring leap years).

10. **Proof by Exhaustion of Cases**

   (a) Prove that the cube of any integer is congruent to $-1, 0$, or 1 modulo 7.

   (b) Show that if n is an integer, then $n^2 \equiv 0, 1, 4 \pmod 5$.

   (c) By checking cases, show that for any integer n, $n^2 - n$ is even.

   (d) Prove by exhaustion of parities that the product of two consecutive integers is even.

11. **Proofs of Equivalence**

   (a) Prove that n is divisible by 2 if and only if n^2 is divisible by 4.

   (b) Prove that $a \equiv b \pmod n$ if and only if a and b leave the same remainder when divided by n.

   (c) Prove that an integer n is divisible by 11 if and only if the alternating sum of its digits is divisible by 11.

   (d) Prove that a and b are both even if and only if $a + b$ is even and $a - b$ is even.

12. **Proof by Induction**

   (a) Find the fallacy in the following proof by **PMI**.
   Theorem Given any collection of n blonde girls. If at least one of the girls has blue eyes, then all n of them have blue eyes.
   Proof: The statement is obviously true for $n = 1$. The step from k to $k + 1$ can be illustrated by going from $n = 3$ to $n = 4$. Assume, therefore, that the statement is true for $n = 3$ and let G_1, G_2, G_3, G_4 be four blonde girls, at least one of which, say G_1, has blue eyes. Taking G_1, G_2, and G_3 together and using the fact that the statement is true when $n = 3$, we find that G_2 and G_3 also have blue eyes. Repeating the process with G_1, G_2 and G_4, we find that G_4 has blue eyes. Thus all four have blue eyes. A similar argument allows us to make the step from k to $k + 1$ in general. □
   Corollary. All blonde girls have blue eyes.
   Proof: Since there exists at least one blonde girl with blue eyes, we can apply the foregoing result to the collection consisting of all blonde girls. □
   Note: This example is from G. Pólya, who suggests that the reader may want to test the validity of the statement by experiment.

   (b) Suban announces to the QRMT class:
   "There will be a surprise test next week. You will not know in advance on which day it will be held."
   One student in the class reasons as follows, by induction on $n = 5 - k$:

   - The test cannot be on Friday (the last working day of the week, $k = 5, n = 0$), because if it hasn't happened before then, the class would know it must be on Friday — and so it would not be a surprise.
   - Similarly, the test cannot be on Thursday, because if it hasn't happened before then, and Friday has already been ruled out, the class would know it must be on Thursday — and so it would not be a surprise.

- Continuing this reasoning backwards, the test cannot be on Wednesday, Tuesday, or Monday either.
- Therefore, the teacher cannot give a surprise test at all!

Yet, when Suban gives the test on Wednesday, the students are indeed surprised. Identify the flaw in the student's reasoning.

(c) Prove by induction that for all integers $n \geq 1$,

$$1^2 + 2^2 + 3^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6}.$$

(d) Show by induction that $7^n - 1$ is divisible by 6 for all integers $n \geq 1$.

(e) Prove that for all $n \geq 1$, $n^3 - n$ is divisible by 6.

(f) Prove that for all integers $n \geq 4$,

$$n! > 2^n.$$

(g) Show that for all integers $n \geq 1$,

$$3^n \geq n^3.$$

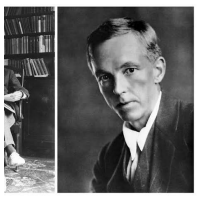
(h) Prove using strong induction that every integer $n > 1$ can be written as a product of prime numbers.

(i) A tiling problem: Prove by strong induction that any $2^n \times 2^n$ chessboard with one square removed can be covered completely by L-shaped trominoes (three-square pieces).

Chapter 6

Primes, GCD, and and the intrigue of Cryptography

"The primes are the jewels in the crown." – Carl Friedrich Gauss



"A mathematician, like a painter or a poet, is a maker of patterns. If his patterns are more permanent than theirs, it is because they are made with ideas. And the primes are the most eternal patterns of all." – G. H. Hardy

"God may not play dice with the universe, but something strange is going on with the prime numbers." – Paul Erdős

The theory of prime numbers and their properties forms the foundation of number theory. At first sight, the study of primes may seem like a purely mathematical pursuit. Yet today prime numbers play a crucial role in modern life, because they underpin cryptographic systems that secure communication over the internet.

In this chapter we introduce the basic ideas of primes and factorisation, methods for computing greatest common divisors, some crucial algorithms and theorems about primes, and finally explain how these notions are used in public-key cryptography, especially the RSA cryptosystem.

6.1 Primes

We have informally discussed primes several times in these notes. We give a formal definition here

Definition 6.1 *A* prime number *is an integer $p > 1$ whose only positive divisors are* 1 *and p itself.*

Numbers greater than 1 that are not prime are called *composite*. $2, 3, 5, 7, 11$ are some examples of primes, while $12 = 3 \times 4$ is composite. In what follows, we develop some properties of prime numbers.

6.1.1 There are infinitely many primes

In Theorem 5.16 we proved there are infinitely many primes. We repeat the theorem here for the sake of completeness.

Theorem 6.1 (Euclid) *There are infinitely many prime numbers.*

Proof: Suppose there are only finitely many: $p_1, \ldots, p_k$. Consider $N = p_1 \cdot p_2 \cdots p_k + 1$. No p_i divides N, since each leaves remainder 1. Thus N is prime or divisible by a new prime, contradicting the assumption[1]. □

6.1.2 Unique prime factorisation

The *Unique Prime Factorisation theorem* (Theorem 5.38) establishes that primes are the "atoms" of arithmetic: every integer factors into primes, and no further. We again repeat it here for completeness.

Theorem 6.2 (Fundamental Theorem of Arithmetic) *Every integer $n > 1$ can be written as a product of primes. This factorisation is unique up to the order of the factors.*

Proof: *Existence:* If n is prime, we are done. Otherwise, assume that the claim is true for all m such that $2 \leq m < n$. Let $n = ab$ with $a, b < n$. By the induction hypothesis, each of a and b has a prime factorisation, hence so does n.

Uniqueness: Suppose

$$n = p_1 p_2 \cdots p_k = q_1 q_2 \cdots q_m$$

with all p_i, q_j prime. Then p_1 divides the right side, hence some q_j. But since q_j is prime, $p_1 = q_j$. Cancelling, we continue inductively to conclude that the multisets $\{p_i\}$ and $\{q_j\}$ are identical. □

6.1.3 The Sieve of Eratosthenes

One of the oldest known algorithms for finding prime numbers is due to *Eratosthenes of Cyrene* (around 200 BCE). The method, called the Sieve of Eratosthenes, systematically eliminates composite numbers from a list, leaving only primes.

> Sift the Two's and Sift the Three's:
> The Sieve of Eratosthenes.
> When the multiples sublime,
> The numbers that remain are Prime.
>
> – Anonymous

[1] Assume you have a finite list of primes. Multiply them together and add 1. Either that number is prime — or you just found a new factor. Either way, you're wrong. Welcome to mathematics!

The Algorithm

To find all primes up to a given number N:

1. Write down the list of integers $2, 3, 4, \ldots, N$.
2. Start with the first number $p = 2$. It is prime.
3. Cross out all multiples of p greater than p itself in the list.
4. Find the next uncrossed number. This is the next prime. Set p to this number and repeat Step 3.
5. Continue until $p^2 > N$. All remaining uncrossed numbers are prime.

Example 6.1 *Primes up to 30*

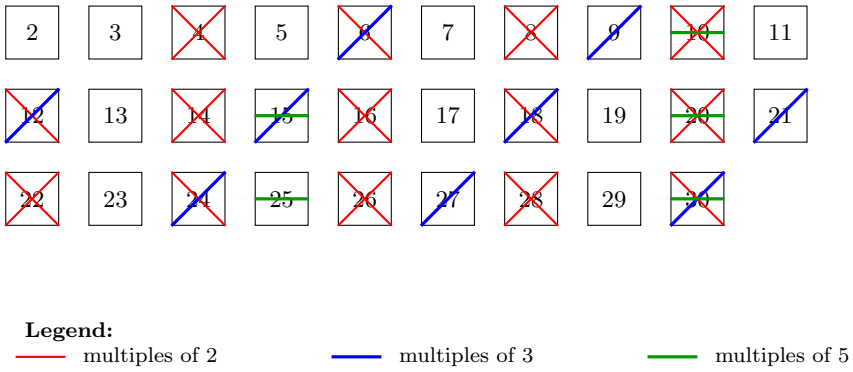


Figure 6.1: Sieve of Eratosthenes up to 30. Uncrossed boxes are primes.

Start with the numbers 2 to 30 (see Figure 6.1).

- 2 is prime. Cross out $4, 6, 8, \ldots, 30$.
- Next uncrossed is 3. Cross out $6, 9, 12, \ldots, 30$.
- Next uncrossed is 5. Cross out $10, 15, 20, 25, 30$.
- Next uncrossed is 7. Since $7^2 = 49 > 30$, we stop.

The remaining numbers are $2, 3, 5, 7, 11, 13, 17, 19, 23, 29$, exactly the primes ≤ 30.

Remarks

- The sieve is simple and efficient for generating primes up to a moderately large bound.
- It takes $O(N \log \log N)$ steps for large N, making it far faster than testing each number individually for divisors.
- This method, though ancient, is still used sometimes[2].

Exercise 6.1 *Find all primes between 2 and 100 using the Sieve of Eratosthenes.*

[2]Unfortunately, this algorithm is too inefficient in practice. The problem of primality testing – how to test whether a given integer is a prime – has been open at least since the times of Euclid and Eratosthenes. The first "efficient" solution for the problem was the AKS algorithm from IIT Kanpur in 2002. In practice, an even faster Miller-Rabin test is used, but it uses probabilistic coin tosses as a primitive, and can give a wrong answer with a very small probability. We will study some of these methods later in these otes

6.1.4 The Prime Number Theorem

The distribution of primes among the natural numbers is highly irregular: sometimes primes cluster close together, other times they are far apart. A natural question, posed already by Gauss and Legendre around the year 1800 when he was a teenager[3], is: *How many primes are there up to a large number n?*

Theorem 6.3 (Prime Number Theorem) *Let $\pi(n)$ denote the number of primes $\leq n$. Then*

$$\pi(n) \sim \frac{n}{\ln n}, \quad \text{as } n \to \infty,$$

meaning that

$$\lim_{n\to\infty} \frac{\pi(n)}{n/\ln n} = 1.$$

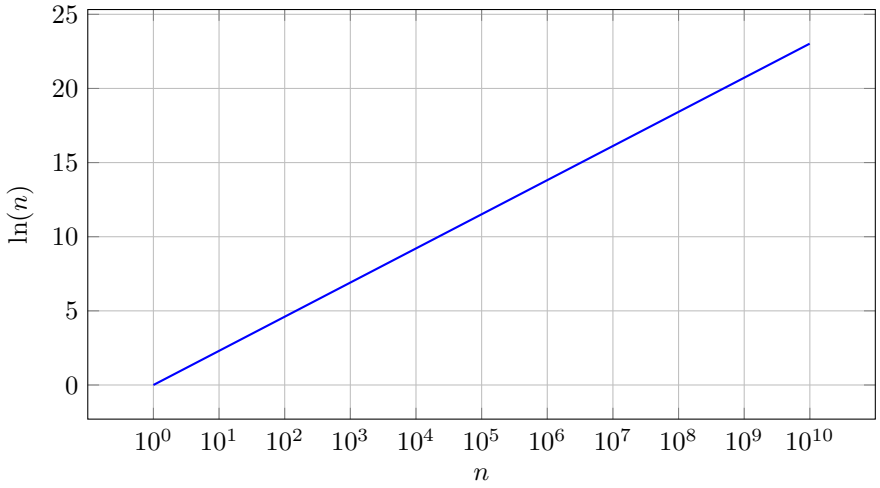


Figure 6.2: The natural logarithm $\ln(n)$ grows very slowly, even as n reaches 10^{10}.

In words: the density of primes around n is roughly $\frac{1}{\ln n}$. The natural logarithm function $\ln n$ is a rather slowly growing function. It can be intuitively understood as a way of answering the question: "How much time does it take to grow from 1 to n, assuming continuous growth at a fixed rate?". See Figure 6.2 for a plot. We will revisit the natural logarithm function later in these notes. So, primes are not very sparse at all[4]. For example, around $n = 10^6$, about one in every 14 numbers is prime.

The proof of the prime number theorem is unfortunately out of the scope of these notes. Interested readers will have to follow up QRMT with some courses on calculus, analysis and algebra for the proof to be accessible.

Exercise 6.2 *Use a calculator to stimate the number of primes between 10^6 and 10^7 using the prime number theorem.*

[3]Historians like to quip: "Even at 15, Gauss preferred to keep his prime secrets private."

[4]Carl Friedrich Gauss (1792–93), as a teenager, investigated prime tables and conjectured the approximation $\pi(n) \approx \mathrm{Li}(n)$, where $\mathrm{Li}(n) = \int_2^n \frac{dt}{\ln t}$ is the logarithmic integral. He recorded this insight in his diary, long before publishing it. Adrien-Marie Legendre independently proposed the formula $\pi(n) \approx \frac{n}{\ln n - 1.08366}$ in 1798. The theorem was finally proved independently in 1896 by Jacques Hadamard and Charles Jean de la Vallée Poussin, using complex analysis and properties of the Riemann zeta function. Their proof showed how deep connections between prime numbers and complex analysis truly are. Interested students will have to wait for a few later course in mathematics to study the proofs

6.1.5 Fermat's Little Theorem

Prime numbers have remarkable properties in modular arithmetic. One of the most beautiful and useful is a result discovered by Pierre de Fermat in 1640, long before the modern development of number theory. It shows that when we raise a number to a large power, the remainder upon division by a prime behaves in a very simple way.

Fermat wrote that if p is a prime and a is not divisible by p, then $a^{p-1} \equiv 1 \pmod{p}$. He gave no proof, and the first rigorous demonstrations came later from Leonhard Euler and others.

This theorem is central in number theory. It lies behind efficient algorithms for primality testing, and it forms one of the key mathematical tools used in public-key cryptography. Before proving it, let us state it precisely.

Theorem 6.4 (Fermat's Little Theorem) *If p is prime and $p \nmid a$, then*

$$a^{p-1} \equiv 1 \pmod{p}.$$

Proof: Multiplication by a permutes the set $\{1, 2, \ldots, p-1\}$ modulo p (why?) Thus

$$a \cdot 2a \cdots (p-1)a \equiv 1 \cdot 2 \cdot \cdots (p-1) \pmod{p}.$$

This gives $a^{p-1}(p-1)! \equiv (p-1)! \pmod{p}$. Since $(p-1)!$ is not divisible by p, we may cancel to obtain $a^{p-1} \equiv 1 \pmod{p}$. □

Exercise 6.3 *Argue that multiplication by a permutes the set $\{1, 2, \ldots, p-1\}$ modulo p. That is, the operation is injective.*

The Fermat's little theorem can be used for primality testing as follows:

Fermat Primality Test Algorithm

1. Given an integer $n > 2$, choose a random integer a where $1 < a < n-1$ and $\gcd(a, n) = 1$.
2. Compute $a^{n-1} \bmod n$.
3. If
$$a^{n-1} \not\equiv 1 \pmod{n},$$
then n is definitely composite.
4. If
$$a^{n-1} \equiv 1 \pmod{n}$$
holds for several randomly chosen values of a, then n is probably prime.

Limitations

- The test is *probabilistic*, not deterministic.
- Certain composite numbers, known as *Fermat pseudoprimes*, can satisfy the congruence for some bases a, causing false positives.
- To reduce error probability, test multiple values of a independently.

Example 6.2 *For $n = 11$:*

$$2^{10} \equiv 1 \pmod{11}, \quad 3^{10} \equiv 1 \pmod{11},$$

so 11 is probably prime.

For a composite number $n = 15$:

$$2^{14} \not\equiv 1 \pmod{15},$$

so 15 is composite.

"The Euclidean algorithm is one of the oldest nontrivial algorithms that has survived to the present day. It can be considered the first efficient algorithm ever devised." – Donald Knuth

6.2 The Greatest Common Divisor

When working with integers, one of the most natural questions is: given two numbers, what is the largest number that divides them both? For example, for 18 and 24 the answer is 6, since 6 is the largest integer that divides both without remainder. This simple idea is called the *greatest common divisor*, or *gcd* for short.

The gcd plays a central role in number theory. It captures the notion of the "common part" of two numbers, and it is fundamental for simplifying fractions, solving equations in integers, and developing modular arithmetic. As we will see, the efficient algorithm to compute gcds, due to Euclid, is one of the oldest and most beautiful algorithms in all of mathematics.

Definition 6.2 *The* greatest common divisor $\gcd(a, b)$ *of two integers* a, b *is the largest integer that divides both.*

Example 6.3

$$\gcd(18, 24) = 6, \qquad \gcd(13, 27) = 1.$$

6.2.1 Euclid's Algorithm

Euclid, over 2000 years ago, discovered a remarkably efficient method, *Euclid_gcd*, for computing $\gcd(a, b)$, $a > 0, b \geq 0$:

$$\begin{array}{ll} Euclid\_gcd(a, b) = a & \text{if } b = 0 \\ Euclid\_gcd(a, b) = Euclid\_gcd(b,\ a \bmod b) & \text{otherwise} \end{array}$$

Repeating this step until the remainder is 0 yields the gcd.

Example 6.4 *To compute* $\gcd(252, 105)$*:*

$$252 = 105 \cdot 2 + 42, \quad Euclid\_gcd(252, 105) = Euclid\_gcd(105, 42),$$

$$105 = 42 \cdot 2 + 21, \quad Euclid\_gcd(105, 42) = Eucid\_gcd(42, 21),$$

$$42 = 21 \cdot 2 + 0 \quad \Rightarrow \quad Euclid\_gcd(21, 0) = 21.$$

Correctness of Euclid's GCD algorithm

To prove the correctness of Euclid's algorithm, we first require the following result which was proved by Euclid.

Claim: If $a = qb + r$, $0 < r < b$, then $gcd(a, b) = gcd(b, r)$

Proof: If $d = gcd(a, b)$ then $d \mid a$ (d divides a) and $d \mid b$ which, in turn, implies that $d \mid (a - qb)$, or $d \mid r$. Thus d is a common divisor of b and r. If c is any common divisor of b and r, then $c \mid (qb + r)$ which implies that $c \mid a$. Thus c is a common divisor of a and b. Since d is the largest divisor of both a and b, it follows that $c \leq d$. It now follows from definition that $d = gcd(b, r)$. □

We can then prove the correctness of *Euclid_gcd* using the principle of mathematical induction.

Theorem 6.5 *For all $b \geq 0$, for all $a > 0$, Euclid_gcd(a, b) computes* gcd(a, b)*, the largest common divisor of a and b.*

Proof:

Basis. $b = 0$. If $b = 0$ then for all $a > 0$, $Euclid\_gcd(a, b) = a = gcd(a, b)$.

Induction hypothesis. For all $b \leq k$ such that $0 \leq b$, for all $a > 0$, $Euclid\_gcd(a, b) = gcd(a, b)$.

Induction step. Consider $b = k + 1$, $a > 0$.

$$\begin{array}{lcll} Euclid\_gcd(a, b) & = & Euclid\_gcd(b, a \bmod b) & \\ & = & gcd(b, a \bmod b) & \text{by the inductive hypothesis} \\ & = & gcd(a, b) & \text{by the Claim above} \end{array}$$

□

Definition 6.3 *Two positive integers are* relatively prime *if their gcd is 1, i.e., they have no non-trivial factors in common.*

6.2.2 The Extended Euclidean Algorithm

Euclid's algorithm is remarkable because it quickly computes the gcd of two numbers. But sometimes we need more than just the gcd: we want to write the gcd as a sum of multiples of the two numbers themselves[5].

For example, for $a = 30$ and $b = 12$, not only is $\gcd(30, 12) = 6$, but also

$$6 = 30 \times (-1) + 12 \times 3.$$

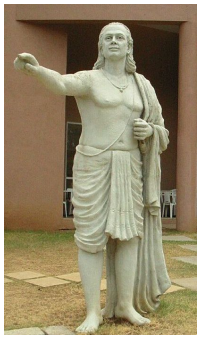
In other words, the gcd can be expressed as a whole-number combination of a and b. This means we can "build" the gcd by taking suitable multiples of 30 and 12. This fact is called Bézout's identity.

The *Extended Euclidean Algorithm* is just a way to find these multipliers at the same time as we compute the gcd.

How it works (idea). We start the same way as in Euclid's algorithm:

$$a = q_1 b + r_1, \quad b = q_2 r_1 + r_2, \quad \ldots$$

until we reach a remainder 0. While we do this, we also keep track of how each remainder is made from a and b. At the end, the last nonzero remainder is the gcd, and the numbers in front of a and b give us the multipliers we want.



Āryabhaṭa's statue in the Inter-University Centre for Astronomy and Astrophysics (IUCAA) at Pune

[5] Euclid tells us what the gcd is; the extended Euclid tells us how to make it.

Example 6.5 *Let's find the gcd of* 26 *and* 7*, and also write it as a combination of* 26 *and* 7*.*

$$26 = 3 \times 7 + 5, \quad 7 = 1 \times 5 + 2, \quad 5 = 2 \times 2 + 1, \quad 2 = 2 \times 1 + 0.$$

So $\gcd(26, 7) = 1$*.*
Now work backwards:

$$1 = 5 - 2 \times 2,$$

$$1 = 5 - 2 \times (7 - 1 \times 5) = 3 \times 5 - 2 \times 7,$$

$$1 = 3 \times (26 - 3 \times 7) - 2 \times 7 = 3 \times 26 - 11 \times 7.$$

So we have

$$1 = 3 \times 26 + (-11) \times 7.$$

This shows that -11 *is a multiplier for* 7 *that makes* $7 \times (-11) \equiv 1 \pmod{26}$*. In other words,* 7 *has an inverse modulo* 26*, and it is* 15 *(since* $-11 \equiv 15 \pmod{26}$*)* [6]*.*

The steps of the general algorithm can then be given as:

General Algorithm (Extended Euclid). Given two integers a and b:

1. If $b = 0$, then $\gcd(a, 0) = a$ and we return $(x, y) = (1, 0)$.
2. Otherwise, divide a by b:
$$a = qb + r, \quad 0 \leq r < b.$$
3. Inductively apply the algorithm to (b, r) to get
$$\gcd(b, r) = xb + yr.$$
4. Substitute $r = a - qb$ to express the gcd in terms of a and b:
$$\gcd(a, b) = ya + (x - qy)b.$$
5. Return $(x', y') = (y,\ x - qy)$.

At the end, we obtain integers x, y such that

$$ax + by = \gcd(a, b).$$

Exercise 6.4 *(Bháskara I, c. 600) "Tell me at once, O mathematician, that number which leaves unity as remainder when divided by any of the numbers from 2 to 6 but is exactly divisible by 7." [Ans : 301]*

Exercise 6.5 *(Mahāvīra, c. 850) "Five heaps of fruits added with two fruits were divided equally between nine travellers; six heaps added with four fruits were divided amongst eight; four heaps increased by one fruit were divided amongst seven. Tell the number of fruits in each heap." [(Least) Ans : 194]*

[6] We had discussed this earlier in Theorem 5.4. The origins of this algorithm can be traced back to Āryabhaṭa (5^{th} c), Jayadeva (c. 10^{th}–11^{th} c.) and later Bháskara II (12^{th} c.). Bháskara in the Līlāvatī and Bījagaṇita gives worked examples of solving linear Diophantine equations with what is recognizably the extended Euclid method.

Why this matters. The extended Euclidean algorithm is very useful:

- It gives the gcd of two numbers.
- It shows how to write the gcd using those two numbers.
- If the gcd is 1, it also finds a modular inverse, which is essential in cryptography.

But how many steps does the Euclidean algorithm take? Is it slow or fast? Amazingly, the answer lies in the famous Fibonacci sequence[7] defined as:

$$\begin{aligned} F_0 &= 0 \\ F_1 &= 1 \\ F_k &= F_{k-1} + F_{k-2}, \text{ for } i \geq 2 \end{aligned}$$

The following theorem establishes the worst-case run rime of the Euclidean algorithm:

Theorem 6.6 *If $a > b \geq 1$ and the invocation Euclid_gcd(a, b) performs $k \geq 1$ repeated calls, then $a \geq F_{k+2}$ and $b \geq F_{k+1}$.*

Exercise 6.6 *Prove the above theorem using the principle of mathematical induction.*

The following result immediately follows:

Theorem 6.7 (Lamé's theorem) *For any integer $k \geq 1$, if $a > b \geq 1$ and $b < F_{k+1}$, then Euclid_gcd(a, b) makes fewer than k repeated calls.*

It is well known that $F_k \leq \phi^{k-1}$ for all $k > 0$, where $\phi = \frac{1+\sqrt{5}}{2}$ is the golden ratio.

Exercise 6.7 *1. Prove that $F_k \leq \phi^{k-1}$ for all $k > 0$ using the principle of mathematical induction.*

2. Argue that if $b < F_{k+1} \leq \phi^k$, then the number of steps required will be approximately $\log b$, which is rather small. So, the algorithm is very efficient.

6.3 Chinese Remainder Theorem

The Chinese Remainder Theorem (CRT) is one of the most beautiful and useful results in number theory. It says that if we know the remainder of a number when divided by several pairwise coprime moduli, then we can uniquely determine the number modulo the product of those moduli.

In other words, instead of working with a single large modulus, we can break a problem into several smaller ones, solve them separately, and then combine the answers back together. This is extremely powerful in computations, because working modulo small numbers is easier.

The theorem was discovered in ancient China, and first appeared in the work of Sun Tzu around the 3rd century AD, who posed problems like: *"Find a number which leaves remainder 2 when divided by 3, remainder 3 when divided by 5, and remainder 2 when divided by 7."* The answer is unique modulo $3 \cdot 5 \cdot 7 = 105$.

The CRT has many applications:

- Efficient computation in modular arithmetic (e.g. fast exponentiation).
- Cryptography, especially RSA, where it is used to speed up decryption.
- Error correction in coding theory.

Thus, the CRT is not only a striking piece of mathematics but also a practical tool.

[7] Though named after Fibonacci, also referred to as the *Leonardo of Pisa*, the sequence was known to ancient Indian mathematicians like Pingala, Bharata muni, Virahanka, Gopala and Hemachandra.

Example 6.6 *An example (Sun Tzu, 3rd century AD).*

"There are certain things whose number is unknown. If we count them by threes, we have two left over; by fives, we have three left over; and by sevens, two are left over. How many things are there?"

Suppose we want to find a number N such that

$$N \equiv 2 \pmod 3, \quad N \equiv 3 \pmod 5, \quad N \equiv 2 \pmod 7.$$

Step 1. The moduli $3, 5, 7$ are pairwise coprime, so the CRT applies. The product is $M = 3 \cdot 5 \cdot 7 = 105$.

Step 2. For each congruence, compute

$$M_1 = \frac{M}{3} = 35, \quad M_2 = \frac{M}{5} = 21, \quad M_3 = \frac{M}{7} = 15.$$

Step 3. Find numbers y_i such that

$$M_1 y_1 \equiv 1 \pmod 3, \quad M_2 y_2 \equiv 1 \pmod 5, \quad M_3 y_3 \equiv 1 \pmod 7.$$

- $35 \equiv 2 \pmod 3$, and $2 \cdot 2 \equiv 1 \pmod 3 \implies y_1 = 2$.
- $21 \equiv 1 \pmod 5$, so $y_2 = 1$ works.
- $15 \equiv 1 \pmod 7$, so $y_3 = 1$ works.

Step 4. Combine everything:

$$N \equiv a_1 M_1 y_1 + a_2 M_2 y_2 + a_3 M_3 y_3 \pmod M,$$

where $(a_1, a_2, a_3) = (2, 3, 2)$.

$$N \equiv 2 \cdot 35 \cdot 2 + 3 \cdot 21 \cdot 1 + 2 \cdot 15 \cdot 1 \pmod{105}.$$

$$N \equiv 140 + 63 + 30 = 233 \pmod{105}.$$

Step 5. Reduce: $233 \equiv 23 \pmod{105}$.

Therefore, the solution is

$$N \equiv 23 \pmod{105}.$$

Indeed, $23 \equiv 2 \pmod 3$, $23 \equiv 3 \pmod 5$, and $23 \equiv 2 \pmod 7$.

Theorem 6.8 (Chinese Remainder Theorem) *Let $n_1, \ldots, n_k$ be pairwise relatively prime (i.e. $\gcd(n_i, n_j) = 1$ for $i \neq j$), and $N = n_1 \cdots n_k$. Then for any integers $a_1, \ldots, a_k$ there is a unique solution $x \pmod N$ to*

$$x \equiv a_i \pmod{n_i}, \qquad i = 1, \ldots, k.$$

Proof: Let $N_i = N/n_i$. Since $\gcd(N_i, n_i) = 1$, there exists y_i with $N_i y_i \equiv 1 \pmod{n_i}$. Then $x = \sum a_i N_i y_i$ solves the congruences. Uniqueness follows: if x and x' are both solutions, then each n_i divides $x - x'$, hence so does N. □

This also has an useful special form:

Theorem 6.9 *Let $n_1, n_2, \ldots, n_k$ be pairwise relatively prime, and let $N = n_1 n_2 \cdots n_k$. Then for all integers x, a,*

$$x \equiv a \pmod{n_i} \text{ for each } i = 1, 2, \ldots, k \quad \iff \quad x \equiv a \pmod N.$$

Proof: ($\Rightarrow$) Assume $x \equiv a \pmod{n_i}$ for each i. Then $n_i \mid (x-a)$ for all i. Since the n_i are pairwise coprime, we claim that

$$n_1 n_2 \cdots n_k \mid (x-a).$$

Thus $x \equiv a \pmod{N}$.

($\Leftarrow$) Conversely, if $x \equiv a \pmod{N}$ then $n \mid (x-a)$. Since $n_i \mid N$, we have $n_i \mid (x-a)$ for each i, i.e. $x \equiv a \pmod{n_i}$ for all i.

This proves the equivalence. □

Exercise 6.8 *(Yih-hiang, c. 700) "Find the number of completed units of work, the same number of units to be performed by each of four sets of 2, 3, 6, 12 workmen, such that after certain whole days' work, there remains 1, 2, 5, 5 units not completed by the respective sets." [(Least) Ans : 17]*

6.4 Public-Key Cryptography

From ancient times to the digital present, cryptography[8] has carried with it an air of mystery and intrigue. In the past, emperors and generals relied on secret codes to protect messages from falling into enemy hands. The Caesar cipher, for instance, shifted letters by a fixed number of places in the alphabet, and even such a simple trick was often enough to change the course of a battle, either way.

But as history has shown, secrecy is never easy to preserve. Every cipher invites the challenge of decryption: a contest between the code maker and the code breaker. This cat-and-mouse game gave rise to ingenious methods on both sides — from the codebreakers at Bletchley Park during World War II, who cracked the German Enigma machine, to modern-day mathematicians working on factoring large numbers.

Public key cryptography is one of the most astonishing turns in this story. Until the 1970s, it was believed that secure communication required two parties to share a secret key in advance. The invention of public key cryptography overturned this assumption: it showed that two people could establish secure communication without ever having met before, using only openly published keys.

This breakthrough was so surprising that it seemed almost paradoxical at first. How can one lock a message with a key that is public, and yet keep it safe from anyone but the intended recipient? The resolution lies in deep number theory, and in particular in the surprising difficulty of problems such as factoring large numbers. It is here that the elegance of mathematics meets the intrigue of secrecy.

In a public key cryptosystem, each participant has

- a **public key** (made widely known),
- a **private key** (kept secret).

The public and the secret keys specify functions that can be applied to any message M which belongs to the set of permissible messages $\mathcal{D}$. The public and secret keys for a participant are a "matched pair" in that they specify functions that are inverses of each other. That is,

$$\begin{aligned} M &= S(P(M)), \\ M &= P(S(M)) \end{aligned}$$

for any $M \in \mathcal{D}$. The challenge in designing a workable public-key cryptosystem is in figuring out how to create a system in which we can reveal a transformation $P()$, without thereby revealing how to compute the inverse transformation $S()$. Public-key cryptography solves this.

6.4.1 How to encrypt?

Suppose Bob wishes to send a message M to Alice, encrypted so that it will look like unintelligible gibberish to an eavesdropper. The procedure for doing so in a public-key cryptosystem is:

[8] See here for an abridged history of cryptography.

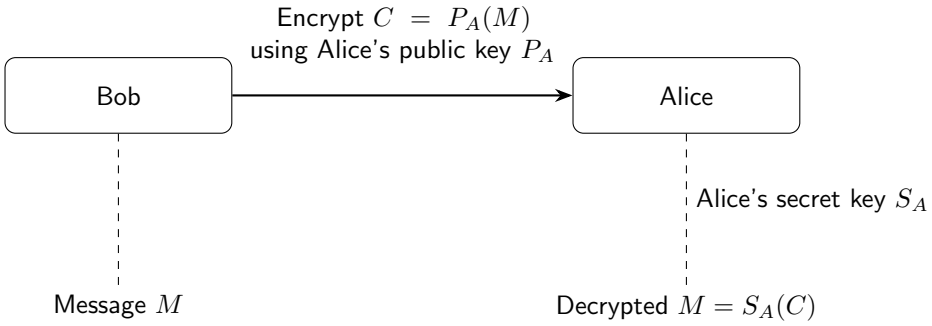


Figure 6.3: Encrypting a message with the receiver's public key.

- Bob obtains Alice's key P_A, from a public directory, or directly from Alice.
- Bob computes the *ciphertext* $C = P_A(M)$, corresponding to the message M and sends C to Alice.
- When Alice receives the ciphertext C, she applies her secret key S_A to retrieve the original message: $M = S_A(C)$.

Since S_A and P_A are inverse functions, Alice can recover M from C. Only Alice is able to compute M from C, as only Alice knows S_A.

6.4.2 Digital signature

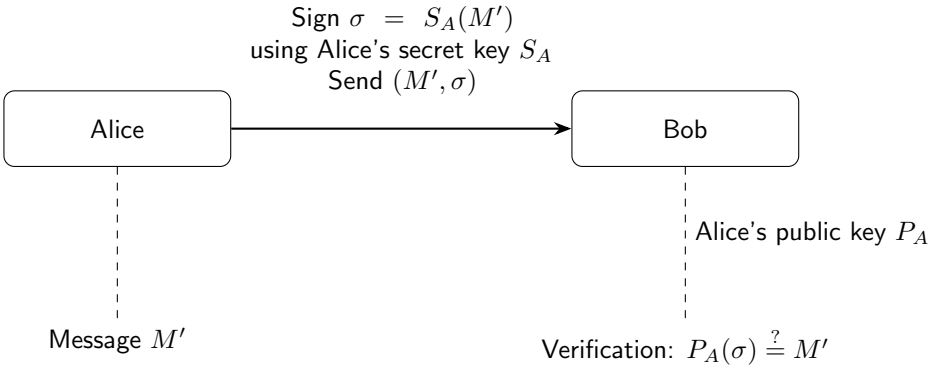


Figure 6.4: Digital signature: Alice signs the message M' with her secret key to produce $\sigma = S_A(M')$. Bob verifies by checking $P_A(\sigma) = M'$ using the public key.

A signature is a device by which the receiver of the signed message can be assured of the identity of the sender and the integrity of the message. Digital signatures are fairly easy to implement in a public-key cryptosystem. Suppose Alice wishes to send Bob a digitally signed response M'. The procedure is as follows:

- Alice computes her *digital signature* σ for the message M' using her secret key S_A and the relation $\sigma = S_A(M')$.
- Alice sends the message-signature pair (M', σ) to Bob.
- When Bob receives (M', σ), he can verify that it came from Alice by using Alice's public key P_A to verify the relation $M' = P_A(\sigma)$. (We presume that M' contains Alice's name, so that Bob knows whose public key to use.) If the equation holds, then Bob concludes that Alice had

sent it. If the equation does not hold, then he concludes that either M' or σ was corrupted by transmission errors, or (M', σ) is an attempted forgery. Note that appending a digital signature necessarily increase the size of the total message.

One can easily combine the two schemes to send signed and encrypted messages.

Exercise 6.9 *Combine the two schemes above to derive a protocol for sending signed and encrypted messages.*

6.5 The RSA Cryptosystem


Ronald Rivest


Adi Shamir


Leonard Adleman

The RSA cryptosystem was proposed by Ronald Rivest, Adi Shamir and Leonard Adleman in 1977. They got the Turing award for their result in 2002.

6.5.1 The RSA procedure

To create his public and secret keys, a person uses the following procedure:

- Select two large prime numbers, p and q, typically of 300 to 600 digits each. To select primes, choose random numbers of this size and test for primality using the Fermat's Little Theorem (Theorem 6.4). By the Prime Number Theorem (Theorem 6.3) we are guaranteed to succeed in just a few trials.
- Compute $n = pq$.
- Select a small odd integer e that is relatively prime to $m = (p-1)(q-1)$. Use Euclid's GCD to test whether a randomly chosen e is relatively prime to m.
- Compute d, the multiplicative inverse of e modulo m. Use the Extended Euclid Algorithm.
- Publish the pair $P = (e, n)$ as his RSA public key.
- Keep secret the pair $S = (d, n)$ as his RSA secret key.

For this scheme, the domain $\mathcal{D}$ is $\mathbb{Z}_n = \{0, 1, \ldots, n-1\}$. That is, the message M must be such that $M \in \mathbb{Z}_n$.The transformation of a message M associated with a public key $P = (e, n)$ is

$$P(M) = M^e \mod n$$

The transformation of the ciphertext C associated with a secret ket $S = (d, n)$ is

$$S(C) = C^d \mod n$$

These equations apply both to encryption and signatures.

6.5.2 Why does it work?

It is easy to see that P and S are inverses:

$$M^{ed} \equiv M \mod n$$

Proof: Note that $m = (p-1)(q-1)$. Since e and d are multiplicative inverses modulo m. We have that

$$ed = 1 + km = 1 + k(p-1)(q-1)$$

Thus if $M \not\equiv 0 \mod p$, we have that

$$\begin{aligned} M^{ed} &\equiv M(M^{p-1})^{k(q-1)} \mod p \\ &\equiv M(1)^{k(q-1)} \mod p \\ &\equiv M \mod p \end{aligned}$$

Note that $M^{p-1} \equiv 1 \mod p$ by Fermat's little theorem (Theorem 6.4).

If $M \equiv 0 \mod p$, we anyway have that

$$M^{ed} \equiv M \mod p$$

Repeating the same argument as above, we have that

$$M^{ed} \equiv M \mod q$$

Hence, by the Chinese Remainder Theorem (Therem 6.8) we have that

$$M^{ed} \equiv M \mod pq \equiv M \mod n$$

□

6.5.3 How secure is RSA?

Multiplying two large primes to form n is easy, but given only n it is extremely difficult to recover p and q with current knowledge. RSA security relies on this asymmetry: the public modulus n reveals nothing practical about the private key. At present, the only known way to obtain d from e and n is to factor n into its prime factors p and q, then compute m and proceed as above. But no one knows how to factor large integers efficiently, despite several centuries of effort. Using known methods, factoring $n = pq$ where p and q are 600-digit primes would require years on today's fastest supercomputers. Until someone comes up with an efficient way to factor, or discovers some other way to compute d from e and n, the system is reasonably secure for all practical purposes.

RSA remains one of the most commonly use public-key encryption system in the digital world, including in banking, stock markets and other financial trasactions.

Summary

- Primes are the building blocks of integers.
- Euclid's algorithm computes gcds efficiently.
- There are infinitely many primes, and every integer has a unique prime factorisation.
- Fermat's little theorem and the CRT are central in modular arithmetic.
- Public-key cryptography allows secure communication without prior key sharing.
- RSA uses primes and modular arithmetic, with security based on the difficulty of factoring.

Problems

1. **Prime numbers and factorisation**
   (a) Prove that there are infinitely many primes of the form $4k + 3$.
   (b) Factor 123456 into primes using successive gcd computations.
   (c) Use the sieve of Eratosthenes to list all primes up to 200. How efficient is this compared to trial division?

2. **Greatest common divisor**
   (a) Compute $\gcd(414, 662)$ using Euclid's algorithm. Show all intermediate steps.
   (b) Use the extended Euclidean algorithm to find integers x, y such that $414x + 662y = \gcd(414, 662)$.

3. **Fermat's little theorem**
   (a) Verify Fermat's little theorem for $a = 7, p = 13$.
   (b) Show by counterexample that Fermat's little theorem does not hold if p is composite.
   (c) Explain how Fermat's little theorem can be used as a primality test. What are its limitations?

4. **Chinese remainder theorem**
   (a) Solve the system of congruences:
   $$x \equiv 2 \pmod 3, \quad x \equiv 3 \pmod 5, \quad x \equiv 2 \pmod 7.$$
   What is the smallest positive solution?
   (b) A recent visitor to Ashoka University, Prof Amartya Kumar Dutta of ISI Kolkata gave the following problems:
      i. In a science camp of 210 students, creativity tests were planned in each of the subjects Mathematics, Physics and Chemistry. The students were to attempt the questions in groups of 5, 6 and 7 respectively. Unfortunately, a few students could not turn up due to unforeseen circumstances and it was found that dividing students into groups of 5, 6 and 7 would leave remainders 2, 1 and 5 respectively. Using kuṭṭaka, a professor of mathematics quickly realised how many students were present and suggested how the students could be grouped if uniformity in number was to be insisted. What was the total number of students present?
      ii. (Hardy-Wright) From Monday, 1 January 2018, six professors begin two-year courses of lectures on Monday, Tuesday, Wednesday, Thursday, Friday, and Saturday. They announce their intention of lecturing at intervals of two, three, four, one, six, and five days respectively. However, no lecture is to take place on a Sunday and a professor would omit his lecture if it falls on a Sunday. For instance, the dates of the lectures of the first professor were January 1,3,5,9,11,13, ... (omitting 7 January, a Sunday). What will be the date of the first Sunday on which all six professors find themselves compelled to omit a lecture?
   (c) Explain why the Chinese Remainder Theorem is crucial for speeding up RSA decryption.

5. **Cryptography and RSA**
   (a) Work out a small RSA example: choose primes $p = 17, q = 23$, construct public/private keys, and encrypt/decrypt the message $M = 42$.
   (b) Suppose Eve can factor the modulus n. Show why RSA becomes insecure.

(c) Consider the Caesar cipher. Implement it with shift $k = 3$, encrypt the message "HELLO", then decrypt it.

(d) Compare the Caesar cipher with RSA in terms of key space, security, and vulnerability to brute-force attack.

6. **Reflection and open-ended problems**

(a) Discuss why factoring large numbers is computationally difficult, and why this difficulty underpins public-key cryptography.

(b) Research the Miller–Rabin primality test. How does it improve upon Fermat's test?

(c) Find a real-world application where digital signatures are used. Explain the role of public and private keys in that context.

Chapter 7

Counting to Beat the Odds

"There is no computation without the art of orderly enumeration." – Paraphrased from Mahāvīra, *Gaṇitasārasaṇgraha* (9th century CE)

"There is no end to what can be counted; the more we count, the more we see patterns emerge." – Paraphrased from the notebooks of Srinivasa Ramanujan

7.1 Introduction

Counting is one of the oldest and most universal human activities. Long before the invention of symbols or written numbers, people used stones, knots, or scratches on bones to keep track of livestock, harvests, or trade. From these humble origins grew the entire edifice of arithmetic, combinatorics, and probability theory.

In modern computer science and mathematics, counting is not merely an act of enumeration but a way of understanding structure. How many possible passwords of a given length can be formed? In how many distinct ways can a set of vertices in a network be connected? In how many ways can certain objects be arranged? Each of these questions is, at heart, a counting problem.

The goal of this chapter is to develop systematic ways of counting the number of possible outcomes in a finite discrete system, and then to use these counts to reason about uncertainty and probability.

- We first learn the *basic counting principles*—the addition and multiplication rules—that form the backbone of all enumeration.
- We then study *permutations* and *combinations*, which tell us how many ways elements can be arranged or selected.
- We then study about the *binomial theorem* and *binomial coefficients*.

- Finally, we build upon these ideas to develop the foundations of *discrete probability*—reasoning about likelihood when outcomes are finite and equally likely.

7.2 The rules of sum and product

7.2.1 The rule of sum

If a task can be done in m ways, and another independent task can be done in n ways, and the two tasks cannot be done simultaneously, then there are $m + n$ ways of choosing one of the two tasks.

Example 7.1 *1. Suppose you may take either a bus or a train to reach a city. If there are 3 bus routes and 2 train routes, then there are* $3 + 2 = 5$ *ways to travel.*

2. Suppose the sections 1, 2 , and 3 of QRMT have 85, 90 and 88 students respectively. If we have to choose one class representative from the combined three sections, then there are $85 + 90 + 88 = 263$ *ways to choose the representative.*

7.2.2 The rule of product

If a task consists of two successive stages, the first of which can be done in m ways and the second in n ways, then the entire task can be done in $m \times n$ ways.

Example 7.2 *1. Suppose you must choose a shirt and a pair of trousers. If there are 4 shirts and 3 trousers, then there are* $4 \times 3 = 12$ *possible outfits.*

2. Suppose the sections 1, 2 , and 3 of QRMT have 85, 90 and 88 students respectively. If we have to choose one class representative from each of the three sections independently, then there are $85 \times 90 \times 88 = 673200$ *ways of choosing the three representatives.*

7.3 Permutations

Definition 7.1 *A* permutation *of a set of n objects is an ordered arrangement of all the objects. The number of permutations of n objects is $n!$, read as "n factorial":*

$$n! = n \cdot (n-1) \cdot (n-2) \cdots 2 \cdot 1.$$

The first can be chosen from n objects, the second from the remaining $n - 1$ objects, the third from the remaining $n - 2$, and so on till 1.

Example 7.3 *How many ways can 5 students line up for a photograph?*
Solution: *There are* $5! = 120$ *possible line-ups.*

Next, consider the number of ways of placing three balls coloured red, blue, and white in 10 boxes numbered $1, 2, 3, \ldots, 10$, if each box can hold only one ball. The red ball can be placed in any of the 10 boxes, the blue ball can be place in any of the 9 remaining boxes, and the white ball can be placed in any of the remaining 8 boxes. Thus the total number of distinct ways to place these balls is $10 \times 9 \times 8 = 720$.

In general, if we have to place r distinctly coloured balls in n distinctly numbered boxes, then the number of distinct ways is

$$n(n-1)(n-2)\cdots(n-r+1) = \frac{n!}{(n-r)!}$$

Example 7.4 *1. In how many ways four examinations can be scheduled in a week so that no two examinations are scheduled on the same way? Considering the days as boxes and the examinations as distinct balls we obtain the result as* $7 \times 6 \times 5 \times 4 = 840$.

2. *How many 3-digit numbers can be formed using the digits* $\{1, 2, 3, 4, 5\}$ *without repetition? Arguing similarly we arrive at* $5 \times 4 \times 3 = 60$.

3. *How many four-digit decimal numbers can be formed without repeating the digits. This according to the above analysis is* $10 \times 9 \times 8 \times 7 = 5040$. *However,* $9 \times 8 \times 7 = 504$ *of them have 0 as the first digit. So, the number of distinct numbers that can be formed is* $5040 - 504 = 4536$. *We can also arrive at this result from the first principles by excluding 0 in the leading position. We then get the number of distinct numbers is* $9 \times 9 \times 8 \times 7 = 4536$.

An equivalent problem of placing distinct balls in boxes is that of arranging distinct objects. By permuting r of n distinct objects we mean to arrange r of these n objects in some order. For example, there are six ways to permute two out of three objects a, b, c. These are ab, ba, ac, ca, bc, cb. Arranging r on n objects amounts to filling r positions with r objects. We get the following definition:

Definition 7.2 Restricted permutations*: If only* r *elements out of* n *distinct elements are to be arranged, the number of permutations is*

$$P(n, r) = \frac{n!}{(n-r)!}.$$

Example 7.5 *How many ways can 3 books be arranged on a shelf from a collection of 10?*

$$P(10, 3) = \frac{10!}{7!} = 720.$$

7.4 Counting functions and power sets

Let us consider the problem of placing 3 distinctly coloured balls into 10 distinctly numbered boxes, but let a box hold as many balls as we wish. Since each of the red, blue, and the white ball can be placed in any of the 10 boxes the total number of ways is

$$10 \times 10 \times 10 = 1000$$

This is exactly the total number of functions that can be defined from the set of balls to the set of boxes.

In general, If there are r distinct balls (the domain has r elements) to be placed in n distinct boxes (the co-domain has n elements) and multiple balls can be mapped on to a box, then the total number of ways is n^r.

Example 7.6 *Let us revisit the problem of scheduling four examinations in a week, but without any restrictions on number of examinations per day. This is exactly the problem of determining the number of functions that can be defined from the courses to days of the week. The result is* $7^4 = 2401$.

Exercise 7.1 *Suppose that the popular tea company serve tea to their customers with 12 types of ingredients, each ingredient can be present or absent. How many different types of tea can they serve?*

Exercise 7.2 *An* n*-bit binary number is defined as* $b_0 + b_1 \times 2^1 + b_2 \times 2^2 + \ldots + b_{n-1} \times 2^{n-1}$*, where each* b_i *is 0 or 1. How many* n*-bit binary numbers are there? How many will have an even number of 1s?*

Exercise 7.3 *How many functions (truth tables) of* n *boolean input and 1 boolean output can we define?*

7.4.1 Power set

The Power set $\mathcal{P}(A)$ of a set A is defined as the set of all subsets of A. For example, the set of all subsets of A is $\mathcal{P}(A) = \{\phi, \{0\}, \{1\}, \{0, 1\}\}$. Suppose the size of A is r. What is the size of $\mathcal{P}(A)$?

Consider the problem of placing each of the r elements in two boxes – inside the subset or outside. Since the 2^r ways of placing the r elements, there are 2^r elements in $\mathcal{P}(A)$.

7.5 Combinations

Consider now the problem of placing three balls, all red, in 10 boxes numbered $1, 2, \ldots, 10$. What are the total number of ways the balls can be placed if each box can hold at most one ball?

If the balls were of three shades of red, and hence distinguishable, then the answer would be $P(10, 3)$. However, if they are indistinguishable then 3! of the ways of their placement actually becomes one, and the number of ways reduces to $P(10, 3)/3!$.

A problem equivalent to placing r indistinguishable objects in n numbered boxes is that of selection of r objects from n distinct objects. If we were to select r objects from n, we can imagine the n objects as boxes, and mark the selected ones.

Definition 7.3 *A* combination *is a selection of objects without regard to order. The number of combinations of r objects chosen from n is*

$$C(n, r) = \frac{n!}{r!(n-r)!}.$$

This is also sometimes denoted as $\binom{n}{r}$.

Example 7.7 *1. Suppose Dining@Ashoka wants to schedule* Tinda Kala Chana *at lunch three times a week. They can clearly do so in $\binom{7}{3} = \frac{7!}{3!4!} = 35$ ways.*

2. From 10 professors, in how many ways can a committee of 3 be chosen? The answer is $\binom{10}{3} = \frac{10!}{3!\,7!} = 120$.

Exercise 7.4 *Clearly $\binom{n}{r} = \binom{n}{n-r}$. Give a combinatorial argument for the above.*

Example 7.8 *Consider a $p \times q$ chess board. In how many ways a rook can move from the bottom-leftmost corner to the top-rightmost, moving only towards right and top? There are total of $p-1$ right moves and $q-1$ top moves to be made. Think of them as red and blue balls respectively. Then, the problem is equivalent to the number of ways of choosing $p-1$ red balls from $p+q-2$ total balls, which is $\binom{p+q-2}{p-1}$. This is also exactly equal to $\binom{p+q-2}{q-1}$.*

Exercise 7.5 *Suppose we have to place r balls in n numbered boxes, where q_1 them are of one colour, q_2 of them are in second colour, ..., and q_k of them are of the k^{th} colour. Argue that the total number of ways to place the balls is*

$$\frac{P(n, r)}{q_1!q_2!\cdots q_k!}$$

Example 7.9 *Among 11 professors there are $C(11, 5) = 462$ ways of selecting a committee of 5 members. Also, there are $C(10, 4) = 210$ ways to select a committee of 5 members so that a particular professor, say professor Rajan, is always included, and there are $C(10, 5) = 252$ ways of selecting a committee of 5 members so that professor Rajan is always excluded.*

We now ask in how many ways we can select a committee of 5 members so that at least one of prof Rajan and prof Suban will be included. The number of selections including both is $C(9, 3) = 84$. The number of selections including Rajan but excluding Suban is $C(9, 4) = 126$, as is the number of selections including Suban but excluding Rajan. Thus, the total number of ways is

$$84 + 126 + 126 = 336$$

Alternatively, since the total number of selections excluding both Rajan and Suban is $C(9, 5)$, the total number of ways of selection is

$$C(11, 5) - C(9, 5) = 462 - 126 = 336$$

7.5.1 Combination with repetitions

Consider the problem of placing r identical balls in n numbered boxes, but allowing as many balls in a box as we wish.

As an equivalent problem, consider the problem of arranging $n+1$ 1s and r 0s with a 1 at the beginning and a 1 at the end of each arrangement. If we consider the 1s as inter-box partitions and 0s as balls, then every such arrangement is a way of placing the r balls in the n boxes with possible repetitions. For example, let $n = 5$, and $r = 4$. Then the sequence

$$1011001101$$

represents placing 1 ball in the first box, none in the second, 2 balls in the third box, none in the fourth, and 1 ball in the fifth box. The number of ways of arranging r 0s and $n+1$ 1s with 1s at both ends is

$$C(n+r-1, r) = \frac{(n+r-1)!}{r!(n-1)!}$$

Example 7.10 *1. The number of ways to select three days in a week with repetitions allowed is* $C(9,3) = 84$.

2. The number of ways to select seven out of three days with repetitions allowed is $C(9,7) = 36$.

3. When three dices are rolled, the number of different outcomes is $C(8,3) = 56$.

7.6 The principle of Inclusion and Exclusion

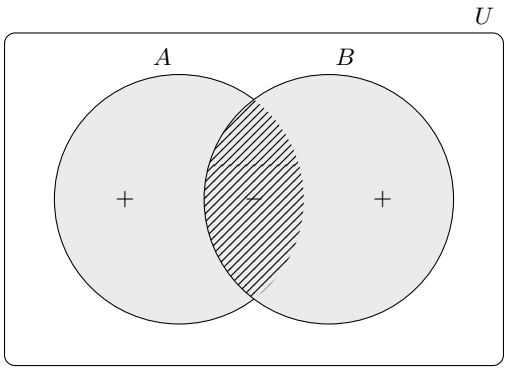


Figure 7.1: The principle of inclusion and exclusion (2 sets).

The problem of Example 7.9 can also be solved by the following method. Among the 462 ways of selecting the committee of 5, let A_1 and A_2 be the set of ways of selection that include Rajan and Suban respectively. Now, since

$$\begin{aligned} &| A_1 | = C(10,4) = 210 \\ &| A_2 | = C(10,4) = 210 \\ &| A_1 \cap A_2 | = C(9,3) = 84 \end{aligned}$$

We. can estimate $| A_1 \cup A_2 |$, the number ways to include at least one of Rajan and Suban, as

$$| A_1 \cup A_2 | = | A_1 | + | A_2 | - | A_1 \cap A_2 | = 210 + 210 - 84 = 336$$

The last subtraction is necessary because the number of ways where both Rajan and Suban are included is double counted in $| A_1 | + | A_2 |$. This method of counting is called the *principle of inclusion and exclusion.*

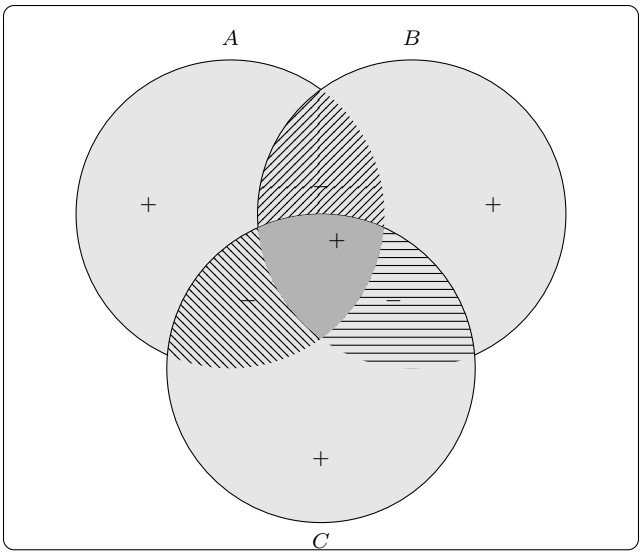


Figure 7.2: The principle of inclusion and exclusion (3 sets).

For any two finite sets A and B,

$$|A \cup B| = |A| + |B| - |A \cap B|.$$

The subtraction corrects for the double counting of elements common to both sets (See Figure 7.1) .

Example 7.11 *Let A be the set of students taking Mathematics (40 students) and B be those taking Physics (30 students), with 10 students taking both. Then, the size of the set taking at least one of Mathematics and Physics is*

$$|A \cup B| = 40 + 30 - 10 = 60.$$

For three finite sets A, B, C,

$$|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C|.$$

Exercise 7.6 *Argue the correctness of the above formula (see Figure 7.2).*

We can generalize the above for n sets $A_1, A_2, \ldots, A_n$ as,

$$\left|\bigcup_{i=1}^{n} A_i\right| = \sum_i |A_i| - \sum_{i<j} |A_i \cap A_j| + \sum_{i<j<k} |A_i \cap A_j \cap A_k| - \cdots + (-1)^{n+1} |A_1 \cap A_2 \cap \cdots \cap A_n|.$$

Example 7.12 (Derangements) *How many permutations of $\{1, 2, 3\}$ have no fixed point (i.e., no element appears in its original position)? Let A_i denote the set of permutations that fix i.*

$$|A_1| = |A_2| = |A_3| = 2, \quad |A_i \cap A_j| = 1, \quad |A_1 \cap A_2 \cap A_3| = 1.$$

Thus,

$$|\overline{A_1 \cup A_2 \cup A_3}| = 3! - (3 \times 2) + (3 \times 1) - 1 = 6 - 6 + 3 - 1 = 2.$$

Hence, there are 2 derangements.

Example 7.13 *Suppose an Ashoka student wants to make up a schedule for a seven-day period during which she will study only one subject each day. She is taking four subjects - Environmental Studies (EVS), Great Books (GB), Economic Politics and Society (EPS) and Quantitative Reasoning*

and Mathematical Thinking (QRMT). Clearly, there are 4^7 different schedules. We want to know the number of schedules that devote at least one day to each subject?

Let A_1 denote the set of schedules in which EVS is never studied; likewise A_2, A_3, A_4 for GB, EPS, and QRMT respectively. Then,

$$A_1 \cup A_2 \cup A_3 \cup A_4$$

is the set of schedules in one or more of the subjects is not included. Since

$$\begin{array}{l} \mid A_1 \mid = \mid A_2 \mid = \mid A_3 \mid = \mid A_4 \mid = 3^7 \\ \mid A_1 \cap A_2 \mid = \mid A_1 \cap A_3 \mid = \mid A_1 \cap A_4 \mid = \mid A_2 \cap A_3 \mid = \mid A_2 \cap A_4 \mid = \mid A_3 \cap A_4 \mid = 2^7 \\ \mid A_1 \cap A_2 \cap A_3 \mid = \mid A_1 \cap A_2 \cap A_4 \mid = \mid A_1 \cap A_3 \cap A_4 \mid = \mid A_3 \cap A_3 \cap A_4 \mid = 1^7 \\ \mid A_1 \cap A_2 \cap A_3 \cap A_4 \mid = 0 \end{array}$$

We obtain

$$\mid A_1 \cup A_2 \cup A_3 \cup A_4 \mid = 4(3^7) - 6(2^7) + 4$$

Consequently, the number of schedules in which all subjects will be included is

$$4^7 - (4(3^7) - 6(2^7) + 4)$$

Exercise 7.7 *Convince yourself $P(7,4) * 4^3$ – there are $P(7,4)$ ways to schedule 4 subjects for 4 of the 7 days, and 4^3 ways to schedule three subjects for the remaining three days – is not the correct answer.*

7.7 Recurrences: Counting by Relations

Consider the following algorithms for sorting a list of integers in ascending order. Suppose we start with a sequence in random order such as

$$\mid 8\ 3\ 5\ 9\ 7$$

The symbol $\mid$ denotes that the elements to the right are unsorted. At the first step we can search through the list for the minimum element, and swap it with the element in the first position. We obtain a sorted part and an unsorted part

$$3 \mid 8\ 5\ 9\ 7$$

Then, keeping the sorted part fixed, we can search through the rest of the list for the minimum element and swap it with the first element in the unsorted part to obtain

$$3\ 5 \mid 8\ 9\ 7$$

Repeating the process we obtain in a sequence

$$3\ 5\ 7 \mid 9\ 8$$

and

$$\begin{array}{c} 3\ 5\ 7\ 8 \mid 9 \\ 3\ 5\ 7\ 8\ 9 \mid \end{array}$$

Our problem is to count the number of comparisons required to sort a random sequence of n elements.

We can do this using the principle of mathematical induction. Let $T(n)$ be the number of comparisons required to solve a sorting problem of size n. If $n = 1$ then the sequence is sorted by default. Otherwise, we need to search through n elements to find the minimum which requires $n-1$ comparisons, and are left with an identical subproblem of size $n-1$ which forms the induction hypothesis $T(n-1)$. Hence,

$$T(n) = \begin{cases} 0 & \text{if } n = 1 \\ T(n-1) + (n-1) & \text{if } n > 1 \end{cases}$$

The above is called a *recurrence relation*. We can solve this by telescoping

$$\begin{aligned} T(n) &= T(n-1) + (n-1) \\ &= T(n-2) + (n-2) + (n-1) \\ &\vdots \quad \vdots \\ &= T(1) + 1 + 2 + \ldots + (n-1) \\ &= 1 + 2 + \ldots + (n-1) = n(n-1)/2 \end{aligned}$$

Thus we require $n(n-1)/2$ comparisons.

A *recurrence relation* defines a sequence by relating each term to earlier ones. Recurrences arise naturally in counting when a choice splits a problem into smaller subproblems. In fact, all counting problems can be expressed through recurrences. Consider the following examples.

Example 7.14 *Consider the counting problem of number of distinct ways of arranging n objects. There is only one way to arrange an empty set of objects, which forms the base case $P(0) = 1$. Suppose $P(n-1)$ is the number of ways of arranging $n-1$ objects for $n > 0$ (the induction hypothesis). Then, the n^{th} object can be inserted in each of the $P(n-1)$ arrangements in n ways. Thus we have*

$$P(n) = \begin{cases} 1 & \text{if } n = 0 \\ n * P(n-1) & \text{if } n > 0 \end{cases}$$

which evaluates to $P(n) = n!$

Example 7.15 *Consider the rook problem of Example 7.8. In how many ways a rook can move from the bottom-leftmost corner to the top-rightmost in a $p \times q$ chessboard, moving only towards right and top? If $p = 1$ or $q = 1$ (a 1-dimensional chessboard), then there clearly is only one way giving us our base case. Otherwise, by the rule of sum, it is the sum of the total number of ways by taking the first step towards the right and the total number of ways by taking the first step towards the top. We obtain*

$$Rook(p,q) = \begin{cases} 1 & \text{if } p = 1 \text{ or } q = 1 \\ Rook(p-1,q) + Rook(p,q-1) & \text{otherwise} \end{cases}$$

Note that $Rook(p-1,q)$ and $Rook(p,q-1)$ are identical subproblems of smaller sizes, and hence are valid induction hypotheses. We saw earlier that this must be equal to $\binom{p+q-2}{p-1} = \binom{p+q-2}{q-1}$.

7.7.1 Pascal's triangle

The fact that the above recurrence evaluates to a combination suggests that combination may itself be expressed as a recurrence. Consider the problem of selecting r identical balls from n boxes without repetition. Clearly, if $r = 0$ or $r = n$ we have exactly one way, giving our bases cases as $C(n,0) = C(n,n) = 1$. Now, by the rule of sum, the total number of ways of selecting is equal to the sum of the total number of ways of selecting by picking one from the n^{th} box and the total number of ways by not picking at all from the n^{th} box. This gives us

$$C(n,r) = \begin{cases} 1 & \text{if } r = 0 \text{ or } r = n \\ C(n-1,r-1) + C(n-1,r) & \text{otherwise} \end{cases}$$

The above recurrence leads us to the famous triangle of numbers called the Pascal's triangle[1].

[1]The triangular arrangement of binomial coefficients is commonly associated with Blaise Pascal, whose posthumous work (1665) presented it systematically. However, the structure was known much earlier. The Italian algebraist Niccolò Tartaglia published its first six rows in 1556. Even earlier, it appeared in the writings of the Chinese mathematician Yàng Huì and in the work of the Persian scholar Omar Khayyām. Consequently, the triangle is known by different names in different traditions: Yàng Huì triangle in China, Khayyām's triangle in Persia, and Tartaglia's triangle in Italy.

Starting with $n = 0$ the triangle is

$$\begin{array}{ccccccc} 1 & & & & & & \\ 1 & 1 & & & & & \\ 1 & 2 & 1 & & & & \\ 1 & 3 & 3 & 1 & & & \\ 1 & 4 & 6 & 4 & 1 & & \\ 1 & 5 & 10 & 10 & 5 & 1 & \\ 1 & 6 & 15 & 20 & 15 & 6 & 1 \end{array}$$

Note that each number $\binom{n}{r}$ can be obtained by adding the two to the top and left in the previous row. Instead of computing $n!$ and $r!$ which are rather inefficient, the triangle gives faster method of computing $\binom{n}{r}$.

Exercise 7.8 *A curious fact about the Pascal's triangle is that is we start from each of the 1s in the first column and add up the numbers along the anti-diagonals (in the direction ↗), then we obtain the Fibonacci numbers. For example*

$$\begin{array}{l} 1 \\ 1 \\ 1+1=2 \\ 1+2=3 \\ 1+3+1=5 \\ 1+4+3=8 \\ 1+5+6+1=13 \\ 1+6+10+4=21 \\ \vdots \end{array}$$

Can you figure out why?

Example 7.16 *A shop keeper in the Asawarpur village adjoining Ashoka university thought of the following problem: how many ways can one make a change of ₹1, given coins of denomination 50p, 25p, 10p, 5p and 1p? More generally, can one write a function to compute the number of ways to change any given amount of money? After giving some thought to the problem she noted that the total number of ways to make change for some amount is equal to the number of ways to make change for the amount without using any of the first kind of coin, plus the number of ways to make change assuming we do use a coin of the first kind. But the latter number is equal to the number of ways to make change for the amount that remains after using one coin of the first kind. With this solution in mind she approached the students of QRMT for a recurrence for the problem. Assume that there is a function $d : \mathbb{N} \to \mathbb{N}$ that given you the denomination of the n^{th} type of coin (for example, $d(1) = 1$, $d(2) = 25$, $d(3) = 10$, $d(4) = 5$ and $d(5) = 50$)*[2].

Exercise 7.9 *Suppose the amount for which change has to be generate is a. Argue that a solution to the counting problem is given by the following recurrence.*

$$Coin(a, n) = \begin{cases} 0 & \text{if } a < 0 \text{ or } n = 0 \\ Coin(a, n-1) + Coin(a - d(n), n) & \text{otherwise} \end{cases}$$

7.8 The Binomial Theorem

When we expand an expression of the form $(x + y)^n$ for a positive integer n, we obtain a sum of products involving powers of x and y. For small n, we can compute these expansions by hand:

$$(x+y)^2 = x^2 + 2xy + y^2, \qquad (x+y)^3 = x^3 + 3x^2y + 3xy^2 + y^3.$$

[2]This is a special case of a general class of problems called the partition problem

A clear pattern emerges: each term consists of a power of x multiplied by a power of y, and the coefficients follow the rows of Pascal's triangle. The binomial theorem makes this pattern precise, expressing $(x+y)^n$ as a sum in which the coefficients are exactly the binomial numbers $\binom{n}{k}$.

Theorem 7.1 (Binomial Theorem) *For any integer $n \geq 0$ and for all commuting symbols or numbers x, y,*

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^{n-k} y^k.$$

Proof: (**A combinatorial proof**) Expand $(x+y)^n$ as a product of n identical factors:

$$(x+y)^n = (x+y)(x+y)\cdots(x+y).$$

A term $x^{n-k}y^k$ is formed by choosing y from exactly k of the n factors and choosing x from the remaining $n-k$ factors. The number of ways to choose which k factors contribute the y is $\binom{n}{k}$. Thus,

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^{n-k} y^k.$$

□

Proof: (**An inductive proof**) We prove the identity by induction on n.
Base case: $n = 0$. $(x+y)^0 = 1$ agrees with the right-hand side.
Inductive step: Assume that for some $n \geq 0$,

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^{n-k} y^k. \qquad (*)$$

We prove the identity for $n+1$.

$$(x+y)^{n+1} = (x+y)(x+y)^n = (x+y)\sum_{k=0}^{n} \binom{n}{k} x^{n-k} y^k.$$

Distribute:

$$(x+y)^{n+1} = \sum_{k=0}^{n} \binom{n}{k} x^{n+1-k} y^k + \sum_{k=0}^{n} \binom{n}{k} x^{n-k} y^{k+1}. \qquad (1)$$

We now *reindex* the second sum. Let $j = k+1$. Then when $k = 0$, $j = 1$, and when $k = n$, $j = n+1$. Also $k = j-1$. Therefore:

$$\sum_{k=0}^{n} \binom{n}{k} x^{n-k} y^{k+1} = \sum_{j=1}^{n+1} \binom{n}{j-1} x^{n-(j-1)} y^j = \sum_{j=1}^{n+1} \binom{n}{j-1} x^{n+1-j} y^j. \qquad (2)$$

Now substitute (2) into (1) and combine the sums by matching powers:

$$(x+y)^{n+1} = \binom{n}{0} x^{n+1} + \sum_{j=1}^{n} \left(\binom{n}{j} + \binom{n}{j-1} \right) x^{n+1-j} y^j + \binom{n}{n} y^{n+1}. \qquad (3)$$

Finally, we apply **Pascal's identity**:

$$\binom{n}{j} + \binom{n}{j-1} = \binom{n+1}{j}.$$

Substituting this into (3) gives:

$$(x+y)^{n+1} = \sum_{j=0}^{n+1} \binom{n+1}{j} x^{n+1-j} y^j.$$

This is exactly the binomial expansion for $(x+y)^{n+1}$, completing the induction.

$$\boxed{(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^{n-k} y^k \text{ for all } n \geq 0.}$$

□

Corollaries

1. $(1+1)^n = \sum_{k=0}^{n} \binom{n}{k} = 2^n$. (Give a purely combinatorial argument for this identity.)
2. $(1-1)^n = \sum_{k=0}^{n} (-1)^k \binom{n}{k} = 0$ for $n \geq 1$.
3. The coefficient of $x^{n-k}y^k$ in $(x+y)^n$ is $\binom{n}{k}$.

Example 7.17 *Using the binomial theorem,*

$$(2x-3)^4 = \sum_{k=0}^{4} \binom{4}{k} (2x)^{4-k}(-3)^k = 16x^4 - 96x^3 + 216x^2 - 216x + 81.$$

7.9 Discrete Probability

Probability provides a mathematical language for reasoning about uncertainty. Everyday situations such as tossing a coin, rolling a die, shuffling cards, choosing passwords, or hashing data involve outcomes that cannot be predicted with certainty in advance. Yet, they are not completely unpredictable: the set of all possible outcomes is known, and many useful questions concern the *likelihood* of particular events. In this section we develop a framework for thinking systematically about such situations.

We begin by describing random experiments and their sample spaces, and how events are formed from outcomes. We then introduce probability measures and the basic axioms that govern them. Next, we discuss random variables, which allow us to assign numerical values to outcomes and reason about distributions, independence, and conditional probabilities. Bayes' theorem provides a powerful tool for updating beliefs in the light of new information. Finally, we conclude with the concepts of information and entropy, which quantify uncertainty and play a central role in modern data science, coding theory, and learning systems.

7.9.1 Experiments, Sample Spaces, and Events

A *random experiment* is a process whose outcome cannot be predicted with certainty in advance, even though all possible outcomes are known. The set of all possible outcomes is called the *sample space*, denoted S. An *event* is a subset $E \subseteq S$.

Example 7.18 *1. Tossing a fair coin:* $S = \{H, T\}$.

2. Rolling a fair die: $S = \{1, 2, 3, 4, 5, 6\}$.

3. Drawing a card from a standard deck of 52 cards.

Two events A and B are *mutually exclusive* if $A \cap B = \varnothing$. A collection of events $A_1, \ldots, A_k$ is *exhaustive* if $A_1 \cup A_2 \cup \cdots \cup A_k = S$.

7.9.2 Probability Measure and Axioms

A *probability measure* assigns to each event $E \subseteq S$ a number $P(E)$ satisfying:

$$P(E) \geq 0, \qquad P(S) = 1, \qquad .$$

When S is finite and all outcomes are equally likely,

$$P(E) = \frac{|E|}{|S|}.$$

Example 7.19 *1. P(even number when rolling a die)* $= \frac{3}{6} = \frac{1}{2}$.

2. *P(ace from a deck)* $= \frac{4}{52} = \frac{1}{13}$.

3. *Consider the problem of dealing a poker hand out of a deck of cards. The sample space consists of $C(52, 5)$ different hands that can be dealt. We assume that all of these are equally probable. What is the probability of getting four aces? We note that 48 possible outcomes from the sample space contain 4 aces (1 way to choose 4 aces and 48 ways to choose the remaining card). So, the probability is $48/C(52, 5) = 0.0000185$.*

4. *What is the probability that in the QRMT class of 88 students, no two have the same birthday (assuming no leap year)? Each of the 88 students can have their birthdays in 365 ways, so the sample space is 365^{88}. Out of these $P(365, 88)$ correspond to the distribution of birthdays so that no two are the same. So, the probability that no two have the same birthday is $P(365, 88)/365^{88} \approx 10^{-5}$, which is rather small. Hence, the probability is close to 1 (almost certain) that at least two people have the same birthday. For how many students is the probability nearly 0.5?*

5. *Suppose 8 Ashoka students are in a tug-of-war team. What is the probability that there are exactly two from each of the four years? Clearly, the sample space is of size 4^8. The total number of ways to select exactly two from each batch is $8!/(2!2!2!2!)$. So the probability is*

$$\frac{8!}{2!2!2!2!4^8} = 0.0385$$

6. *Consider the experiment of shooting a target until there is a hit where the probability of a hit or a miss in a single shot is 1/2 (not a very good shooter). The, the probability of k misses before a hit is $2^{-(k+1)}$. Let A denote the event that there is a hit in no more than 5 misses. Then $A = \{h, mh, mmh, mmmh, mmmmh, mmmmmh\}$. Thus*

$$P(A) = \sum_{k=0}^{5} 2^{-(k+1)} = 0.984$$

Let B denote the event that there is a hit after odd number of misses. Then[3]*,*

$$P(B) = \sum_{i=1}^{\infty} 2^{-2i} = 1/3$$

On the other hand, if C is the event that there is a hit after an even number of misses, then

$$P(C) = \sum_{i=1}^{\infty} 2^{-2i+1} = 2/3$$

[3]We will study the methods to evaluate such infinite summations in Section 9.1.

7.9.3 Random Variables, probability distribution, expectation and variance

A *random variable* is a function

$$X : S \longrightarrow \mathbb{R}.$$

It assigns a numerical value to each outcome of the experiment.

Example 7.20 *If $S = \{1, 2, 3, 4, 5, 6\}$, then $X(\omega) = \omega$ simply records the number rolled.*

The *distribution* of X is given by

$$P(X = x) = |\{\omega \in S : X(\omega) = x\}|/|S|.$$

The *expectation* or *expected value* of a discrete random variable X is the weighted average of its possible values, where each value is weighted by its probability. It captures the *long-run average* or the *mean* outcome if the experiment is repeated many times.

Let X take values $x_1, x_2, \ldots, x_n$ with probabilities $P(X = x_i)$, or more generally let X take countably many values. Then the expectation of X is defined as

$$\mathbb{E}[X] = \sum_x x\, P(X = x).$$

Interpretation. If we repeat the experiment many times and record the values of X, the average of the recorded values will be close to $\mathbb{E}[X]$. Thus, expectation represents the *typical* or *average* outcome in the long run.

Example 7.21 1. ***Fair Die.*** *The random variable X equals the number rolled:*

$$\mathbb{E}[X] = \frac{1}{6}(1 + 2 + 3 + 4 + 5 + 6) = \frac{21}{6} = 3.5.$$

The expected value need not be a possible outcome.

2. ***Indicator Variables.*** *If I_A is the indicator of event A (i.e., $I_A = 1$ if A occurs and 0 otherwise), then*

$$\mathbb{E}[I_A] = P(A).$$

So expectations generalize probabilities.

Example 7.22 *Consider rolling two fair six-sided dice. Let X be the value on the first die and Y the value on the second die. Define the random variable*

$$S = X + Y,$$

the sum *of the two dice.*

Since each die has 6 faces and the dice are independent, the sample space consists of 36 *equally likely ordered pairs:*

$$S = \{(1,1), (1,2), \ldots, (6,6)\}, \qquad P((i,j)) = \frac{1}{36}.$$

The possible values of the sum S range from 2 *to* 12*. To compute $P(S = s)$, we count how many ordered pairs (i, j) satisfy $i + j = s$.*

$$P(S = s) = \frac{|\ \{(i,j) :\ i + j = s\}\ |}{36}.$$

The number of such pairs increases from $s = 2$ up to $s = 7$, then decreases symmetrically:

$$P(S = s) = \begin{cases} \frac{s-1}{36}, & 2 \le s \le 7, \\ \frac{13-s}{36}, & 8 \le s \le 12. \end{cases}$$

Distribution Table.

s	2	3	4	5	6	7	8	9	10	11	12
number of pairs (i,j)	1	2	3	4	5	6	5	4	3	2	1
$P(S=s)$	$\frac{1}{36}$	$\frac{2}{36}$	$\frac{3}{36}$	$\frac{4}{36}$	$\frac{5}{36}$	$\frac{6}{36}$	$\frac{5}{36}$	$\frac{4}{36}$	$\frac{3}{36}$	$\frac{2}{36}$	$\frac{1}{36}$

Explanation. *The value $S=7$ is the most likely because there are more pairs of dice that add to 7 than to any other sum:*

$$(1,6),(2,5),(3,4),(4,3),(5,2),(6,1).$$

There are 6 such pairs, giving $P(S=7)=6/36=1/6$.

The distribution is symmetric because (i,j) and (j,i) contribute to sums equally. Values near the center have the largest number of combinations, while values near the extremes have fewer.

Example 7.23 *Suppose we repeat an experiment consisting of independent identical trials, each of which results in a* success *with probability p and a* failure *with probability $1-p$, where $0<p\leq 1$.*

Let T be the random variable defined as:

$$T = \textit{the number of trials until the first success occurs.}$$

Thus T takes values in $\{1,2,3,\dots\}$.

Probability Distribution. *For the first success to occur on trial k, the first $k-1$ trials must be failures (each with probability $1-p$), and the k-th trial must be a success.*

$$P(T=k)=(1-p)^{k-1}\,p, \qquad k=1,2,3,\dots$$

This distribution is called the Geometric distribution *with parameter p.*

Verification. *The probabilities sum to 1*[4]*:*

$$\sum_{k=1}^{\infty}(1-p)^{k-1}p = p\sum_{k=0}^{\infty}(1-p)^{k} = p\cdot\frac{1}{1-(1-p)}=1.$$

Expected Value.

$$\mathbb{E}[T]=\sum_{k=1}^{\infty}k\,(1-p)^{k-1}p=\frac{1}{p}.$$

Thus, the average number of trials needed to get a success is $\dfrac{1}{p}$.

Example. *If the probability of success on each trial is $p=\frac{1}{4}$ (for example, getting a four when rolling a die), then*

$$\mathbb{E}[T]=\frac{1}{p}=4.$$

On average, one must roll the die 4 times to get the first four.

While the expectation of a random variable measures its average value, the *variance* measures how much the values of the random variable tend to spread out around the mean.

Let X be a discrete random variable with expectation $\mu=\mathbb{E}[X]$. The *variance* of X is defined as

$$\operatorname{Var}(X)=\mathbb{E}\big[(X-\mu)^2\big].$$

The quantity $(X-\mu)^2$ measures the squared deviation of X from its mean, and the expectation averages this deviation over all outcomes.

[4]We will study the methods to evaluate such infinite summations in Section 9.1.

The square root of the variance is called the *standard deviation*:

$$\sigma = \sqrt{\mathrm{Var}(X)}.$$

Often it is easier to compute variance using the identity

$$\mathrm{Var}(X) = E[X^2] - (E[X])^2.$$

This follows by expanding $(X - \mu)^2 = X^2 - 2\mu X + \mu^2$ and taking expectation.

7.9.4 The Bernoulli Distribution

Introduction

Many experiments in probability have only two possible outcomes — such as *success/failure*, *yes/no*, or *head/tail*. The simplest random variable that models such an experiment is called a *Bernoulli random variable*.

Definition

A random variable X is said to have a *Bernoulli distribution* with parameter p (where $0 \leq p \leq 1$) if

$$P(X = 1) = p \quad \text{and} \quad P(X = 0) = 1 - p.$$

We write this as

$$X \sim \mathrm{Bernoulli}(p).$$

Here $X = 1$ represents "success" and $X = 0$ represents "failure".

Probability Mass Function

The probability mass function (PMF) of X can be compactly expressed as

$$P(X = x) = p^x (1 - p)^{1-x}, \qquad x \in \{0, 1\}.$$

Expectation and Variance

The *expected value* (mean) and *variance* of a Bernoulli random variable are

$$E[X] = p, \qquad \mathrm{Var}(X) = p(1 - p).$$

Proof:

$$E[X] = (0)(1 - p) + (1)(p) = p,$$

and

$$E[X^2] = (0^2)(1 - p) + (1^2)(p) = p.$$

Therefore

$$\mathrm{Var}(X) = E[X^2] - (E[X])^2 = p - p^2 = p(1 - p).$$

Example

Consider a fair coin toss where success is "getting a head". Then $p = \frac{1}{2}$ and

$$P(X = 1) = P(X = 0) = \tfrac{1}{2}.$$

Hence $E[X] = \frac{1}{2}$ and $\mathrm{Var}(X) = \frac{1}{4}$.

Remarks

- The Bernoulli distribution is the simplest discrete distribution.
- The sum of n independent Bernoulli(p) random variables follows a *Binomial* distribution Binomial(n, p).
- Many random processes—such as coin tossing, system reliability, and binary classification—are modeled as Bernoulli trials.

7.9.5 Independence

Two events A and B are independent if

$$P(A \cap B) = P(A)P(B)$$

Two random variables X and Y are independent if for all x, y,

$$P(X = x, Y = y) = P(X = x)\, P(Y = y).$$

Example 7.24 *1. Consider two independent coin tosses. The sample space is* $S = \{hh, ht, th, tt\}$. *If* $P(h) = P(t) = 1/2$ *for each toss, then* $P(hh) = P(ht) = P(th) = P(tt) = 1/4$.

2. Let $X =$ *result of die 1,* $Y =$ *result of die 2. Then* X *and* Y *are independent:*

$$P(X = 4, Y = 2) = \frac{1}{6} \cdot \frac{1}{6}.$$

3. Roll two fair dice. Let

$$A = \{\textit{the first die shows a } 4\}, \qquad B = \{\textit{the second die shows an even number}\}.$$

Then

$$P(A) = \frac{1}{6}, \qquad P(B) = \frac{1}{2}.$$

Since the dice are independent,

$$P(A \cap B) = P(\textit{first die is } 4 \textit{ and second die is even}) = \frac{1}{6} \cdot \frac{1}{2} = \frac{1}{12}.$$

4. Select one card uniformly from a standard 52-card deck. Let

$$C = \{\textit{the card is red}\}, \qquad D = \{\textit{the card is a Heart}\}.$$

Then:

$$P(C) = \frac{26}{52} = \frac{1}{2}, \qquad P(D) = \frac{13}{52} = \frac{1}{4}.$$

However,

$$P(C \cap D) = P(\textit{the card is a red Heart}) = \frac{13}{52} = \frac{1}{4}.$$

But

$$P(C)\, P(D) = \frac{1}{2} \cdot \frac{1}{4} = \frac{1}{8}.$$

The events C *and* D *are* not *independent. Knowing that the card is a Heart* already implies *that it is red. So the occurrence of* D *changes the probability of* C.

7.9.6 Conditional Probability

In many situations, additional information is available about the outcome of an experiment. Such information can change the likelihood of events. The idea of *conditional probability* allows us to update probabilities when we learn that some event has already occurred.

If $P(B) > 0$, $P(A \mid B)$ is the probability of A given that B has occurred is given as

$$P(A \mid B) = \frac{P(A \cap B)}{P(B)}.$$

Example 7.25 *Let $A =$ "card is a King", $B =$ "card is a face card". Then*

$$P(A \mid B) = \frac{4/52}{12/52} = \frac{1}{3}.$$

Bayes' Theorem

$$P(A \mid B) = \frac{P(B \mid A)P(A)}{P(B)}.$$

More generally, if $\{A_1, \ldots, A_k\}$ partitions S,

$$P(A_i \mid B) = \frac{P(B \mid A_i)P(A_i)}{\sum_{j=1}^{k} P(B \mid A_j)P(A_j)}.$$

Example 7.26

1. *A disease has a probability $P(D) = 0.01$. A test has a probability $P(\textit{positive} \mid D) = 0.99$ and $P(\textit{positive} \mid \overline{D}) = 0.05$.*

$$P(D \mid \textit{positive}) = \frac{0.99 \cdot 0.01}{0.99 \cdot 0.01 + 0.05 \cdot 0.99} \approx 0.166.$$

2. *A coin was chosen at random and tossed. The probability that a fair coin was chosen and head shows is 1/3. The probability that a fair coin was chosen and tail shows is also 1/3. The probability that an unfair coin was chosen and head shows is 1/12. The probability that an unfair coin was chosen and head shows is 1/4.*

   Clearly, the probability that head shows is

$$\frac{1}{3} + \frac{1}{12} = \frac{5}{12}$$

   and the probability that an unfair coin was chosen is

$$\frac{1}{12} + \frac{1}{4} = \frac{1}{3}$$

   Thus, the conditional probability that an unfair coin was chosen given that head shows is

$$\frac{1/12}{5/12} = \frac{1}{5}$$

   and the conditional probability that head shows given an unfair coin was chosen is

$$\frac{1/12}{1/3} = \frac{1}{4}$$

3. *Three dice were rolled. Given that no two faces were the same, what is the probability that there was an ace? Let A denote the event that there was an ace, and B the event that no two faces were the same. Note that*

$$P(B) = \frac{P(6,3)}{6^3} \qquad P(A \cap B) = \frac{3P(5,2)}{6^3}$$

   Hence,

$$P(A \mid B) = \frac{3P(5,2)}{P(6,3)} = \frac{1}{2}$$

Problems

1. A department offers 6 foundation courses, 8 elective courses, and 3 project courses.
   (a) In how many ways can a student choose exactly one course of any type?
   (b) In how many ways can a student choose one foundation and one elective course?

2. A password consists of either:
   - any string of 4 lowercase letters, or
   - any string of 3 digits.

   How many possible passwords are there?

3. A student rolls a fair six-sided die until a 6 is obtained. How many different length-n outcome sequences are possible? (Hint: sequences end with a 6.)

4. How many permutations of the letters A, B, C, D, E begin with a consonant and end with a vowel?

5. In how many ways can 10 runners finish a race if ties are not allowed? How many ways if the top 3 places matter but the remaining order does not?

6. How many functions are there from an n-element set to a k-element set? How many of these functions are injective?

7. Show that a set of size n has 2^n subsets. (Hint: Use the binary string representation or apply the product rule.)

8. Prove that $\binom{n}{k} = \binom{n}{n-k}$ using
   (a) a bijection between k-subsets and $(n-k)$-subsets,
   (b) counting of bitstrings of length n with k ones.

9. A class of 40 students must form project teams of size 3. How many different teams are possible? How many different collections of *five disjoint* teams of size 3 can be formed?

10. In a group of 120 students,

    $$|A| = 50, \quad |B| = 60, \quad |C| = 40, \quad |A \cap B| = 20, \ |A \cap C| = 15, \ |B \cap C| = 18, \ |A \cap B \cap C| = 8.$$

    How many students take at least one of the three courses?

11. How many integers between 1 and 1000 are divisible by at least one of 2, 3, or 5?

12. Prove the general n-set inclusion–exclusion formula. (Hint: Count how many times each element is included/excluded.)

13. Let a_n be the number of binary strings of length n with no two consecutive 1s.
    (a) Derive a recurrence for a_n.
    (b) Compute a_1, a_2, a_3, a_4, a_5.
    (c) Show that the recurrence is the Fibonacci recurrence.

14. Let D_n be the number of derangements of n items. Using the argument from class, show that

    $$D_n = (n-1)(D_{n-1} + D_{n-2}).$$

15. Using the combinatorial argument "subsets that contain n and subsets that do not," prove Pascal's identity
$$\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}.$$

16. **Binomial theorem.** Expand $(x+y)^6$ and determine the coefficient of x^4y^2 and of x^2y^4. Verify using $\binom{6}{2} = \binom{6}{4}$.

17. Show that
$$\sum_{k=0}^{n} \binom{n}{k}^2 = \binom{2n}{n}.$$
(Hint: Interpret both sides combinatorially.)

18. Each student in a class of n students writes down the last digit of their phone number (an integer from 0 to 9, assumed equally likely).
    (a) What is the probability that all n students have *different* last digits? (Express the answer as a product.)
    (b) Evaluate this probability for $n = 5$ and for $n = 10$.
    (c) For what smallest n is this probability less than 0.5?

19. How many 5-card poker hands are possible? How many contain at most four aces?

20. How many subsets of 1,2,...,nhave an even number of elements? (Show your reasoning.)

21. Roll two fair dice.
    (a) Write down the sample space explicitly.
    (b) Compute the distribution of the sum.
    (c) Compute $P(\text{sum is even})$, $P(\text{sum is a prime})$, and $P(\text{sum} = 10)$.

22. Let X be the number of heads in 4 tosses of a fair coin. Compute $\mathbb{E}[X]$ (a) from the definition, (b) using linearity of expectation.

23. For the same random variable X, compute $\mathrm{Var}(X)$ using
$$\mathrm{Var}(X) = \mathbb{E}[X^2] - \mathbb{E}[X]^2.$$

24. A fair die is rolled twice. Let X be the maximum of the two outcomes. Find the probability distribution of X and compute $\mathbb{E}[X]$.

25. Two fair dice are rolled. Let A be the event "sum is 7" and B be the event "first die is even." Are A and B independent? Justify carefully.

26. You pick a card from a well-shuffled deck. Let A be the event "the card is red," and B be "the card is a king." Are A and B independent?

27. In a class, 40% of students play basketball, 30% play football, and 15% play both. Compute $P(\text{basketball} \mid \text{football})$.

28. Three cards are drawn without replacement from a standard deck. Compute the probability that all three are face cards given that the first is a face card.

29. A spam-filtering system flags emails as "spam" or "not spam." Suppose:
$$P(\text{email is spam}) = 0.20,$$
$$P(\text{flagged as spam} \mid \text{spam}) = 0.95, \qquad P(\text{flagged as spam} \mid \text{not spam}) = 0.05.$$
If an incoming email is flagged as spam, compute the probability that it is actually spam.

30. Let $X \sim \text{Bernoulli}(p)$. Compute the distribution, expectation, and variance of

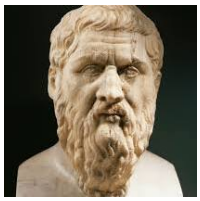
$$S_n = X_1 + X_2 + \cdots + X_n,$$

where the X_i are independent Bernoulli(p) random variables.

Chapter 8

Even infinity admits a hierarchy we must confront: The Real Numbers

"God created the integers; all else is the work of man" – Leopold Kronecker



"He is unworthy of the name of man who is ignorant of the fact that the diagonal of a square is incommensurable with its side" – Plato

The real numbers are the backbone of modern mathematics. They unify discrete and continuous quantities, allow us to measure lengths, areas, speeds, and forces, and they form the natural domain for calculus, physics, probability, and geometry. Yet the construction of the real numbers is not obvious, and their necessity arises from deep conceptual difficulties that puzzled mathematicians for millennia.

In this chapter we try to build a conceptual and computational foundation for understanding the real numbers, their structure, and the functions defined on them.

8.1 Why Do We Need Real Numbers?

As we saw earlier in these notes, mathematics begins with the natural numbers $\mathbb{N}$, which suffice for counting and simple arithmetic. But subtraction motivates the integers $\mathbb{Z}$, division motivates the rationals $\mathbb{Q}$, and geometry quickly reveals that even rational numbers are insufficient.

8.1.1 Incompleteness of the Rationals and the irrational persecution of Hippasus

In Theorem 5.13 we discussed a classical proof that shows that $\sqrt{2} \notin \mathbb{Q}$. This was shocking in antiquity and led to the development of more sophisticated number systems. Thus geometry forces us beyond the rationals.

> "He is unworthy of the name of man who is ignorant of the fact that the diagonal of a square is incommensurable with its side" – (Plato)

Hippasus is often credited with the discovery of irrational numbers. According to common folklore the Pythagoreans was rather upset with this discovery, and Hippasus is supposed to have drowned at sea, apparently as a punishment from the gods for divulging this troublesome fact.

There are more reason as to why rational are incomplete. Although $\mathbb{Q}$ is dense on the line, it contains "holes". A sequence of rational numbers may converge to a limit that is not rational.

Consider, for example,

$$y_n = \sum_{k=1}^{n} 10^{-k!}$$

Each y_n is rational, but the limit is the Liouville constant, which is not rational.

Theorem 8.1 *The Liouville constant*

$$L = \sum_{k=1}^{\infty} \frac{1}{10^{k!}} = \frac{1}{10} + \frac{1}{10^2} + \frac{1}{10^6} + \frac{1}{10^{24}} + \cdots$$

is irrational.

Proof: We will use the fact that a decimal number is rational if and only if its decimal expansion is *eventually periodic*; that is, after some point its digits repeat with a fixed period. We will prove this in a exercise problem at the end of this chapter.

Let us analyse the decimal expansion of L. The term $\frac{1}{10^{k!}}$ contributes a single '1' in the $k!$-th decimal place and zeros elsewhere. Thus L has a '1' in the decimal positions

$$1!,\ 2!,\ 3!,\ 4!,\ \ldots$$

and all other digits are 0. Therefore the distances between successive positions of '1' are

$$\begin{aligned} 2! - 1! &= 1 \\ 3! - 2! &= 4 \\ 4! - 3! &= 18 \\ 5! - 4! &= 96 \end{aligned}$$

and in general

$$(k+1)! - k! = k! \cdot k$$

These gaps grow without bound as $k \to \infty$.

Now suppose, for the sake of contradiction, that L is rational. Then its decimal expansion would be eventually periodic: there exist integers $N \geq 1$ and $p \geq 1$ such that all digits from position N onward repeat with period p. In particular, after position N, if a digit '1' occurs once, then the same digit '1' must occur again exactly p places later, and then again p places after that, and so on. Thus the distances between successive '1's beyond N must all be *exactly* p.

However, choose k large enough that $k! \cdot k > p$ and also $k! > N$. Then between the '1' at position $k!$ and the next '1' at position $(k+1)!$ there are

$$(k+1)! - k! = k! \cdot k$$

zeros — a gap strictly greater than p. This contradicts the assumption that the decimal expansion is eventually periodic with period p.

Hence no such period p can exist, and the decimal expansion of L is not eventually periodic. Therefore L is irrational. □

Hence, rhe rational numbers are not complete. Limits of rational sequences need not be rational. To properly define limits, infinite series, areas, lengths, and solutions of differential equations, we need a larger number system.

8.1.2 Zeno's paradox and infinite processes: Achilles and the Tortoise

Zeno of Elea (5th century BCE) presented several paradoxes to argue that motion is impossible. His most famous example is the paradox of *Achilles and the Tortoise*, which troubled mathematicians and philosophers for centuries. We now explain the paradox in detail and show how it naturally leads to the concept of limits and infinite series.

The Setup

Suppose Achilles, the fastest of the Greek heroes, runs ten times faster than a tortoise. The tortoise is given a head start of D metres. Common sense says Achilles will quickly overtake the tortoise. Zeno's reasoning, however, appeared to show otherwise.

Zeno's Argument

Zeno argued as follows:

1. Achilles first runs to the tortoise's starting point (distance D).
2. During that time, the tortoise moves ahead a little.
3. Achilles then runs to this new position.
4. Again the tortoise moves slightly ahead.
5. This continues forever, *ad infinitum*.

Thus Achilles must complete *infinitely many tasks*: first reach D, then $D(1+r)$, then $D(1+r+r^2)$, and so on, where r is the ratio of the tortoise's speed to Achilles' speed. Zeno concluded that Achilles can never overtake the tortoise.

Mathematical Formulation

Let v_A be Achilles' speed and v_T be the tortoise's speed, with $v_A > v_T$. Let the tortoise's head start be D.

The time Achilles takes to reach the tortoise's initial position is

$$t_1 = \frac{D}{v_A}.$$

During this time the tortoise moves

$$D_1 = v_T t_1 = D \cdot \frac{v_T}{v_A}.$$

Next, Achilles takes

$$t_2 = \frac{D_1}{v_A} = \frac{D}{v_A}\left(\frac{v_T}{v_A}\right)$$

to reach that point; during this period the tortoise moves

$$D_2 = D\left(\frac{v_T}{v_A}\right)^2.$$

Continuing, we obtain an infinite sequence of positive times

$$t_1,\ t_2,\ t_3,\ \ldots$$

with

$$t_n = \frac{D}{v_A}\left(\frac{v_T}{v_A}\right)^{n-1}.$$

Thus the total time required for Achilles to reach all points previously occupied by the tortoise is

$$T = t_1 + t_2 + t_3 + \cdots = \frac{D}{v_A}\left(1 + r + r^2 + r^3 + \cdots\right),$$

where

$$r = \frac{v_T}{v_A} \quad \text{and} \quad 0 < r < 1.$$

Resolution via convergent geometric series

We will show in Section 9.1 that the infinite geometric series with $0 < r < 1$ satisfies

$$1 + r + r^2 + r^3 + \cdots = \frac{1}{1-r}.$$

Hence

$$T = \frac{D}{v_A} \cdot \frac{1}{1-r} = \frac{D}{v_A - v_T}.$$

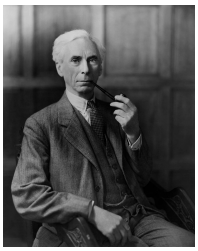
That is, the infinite sum actually converges to a finite quantity, a rational number if r is rational. This is the standard "relative speed" formula for the time taken for a faster runner to catch a slower one. The total time T is finite.

Achilles does indeed overtake the tortoise.

What Zeno misunderstood

Zeno implicitly assumed that a process involving infinitely many steps must take an infinite amount of time. Modern mathematics shows that an infinite series of positive quantities may converge to a finite sum. Here the quantities t_n become smaller so rapidly that their infinite sum remains finite.

This is only made rigorous in the *real numbers*, which allow us to define limits and convergent series. In this sense, Zeno's paradox hints at the need for a complete number system.



"The paradoxes of Zeno the Eleatic and the difficulties in the analysis of space, of time, and of motion, are all completely explained by means of the modern theory of continuity. This is because a non-contradictory theory has been found, according to which the continuum is composed of an infinity of distinct elements; and this formerly appeared impossible" – Bertrand Russell

Understanding the resolution of Zeno's paradox requires understanding limits, convergence, and the structure of the real numbers. The resolution uses *infinite series* and the idea of a *limit*. These require a complete number system — the real numbers. A follow up course on *Calculus* is strongly recommended.

"It is by the idea of limit alone that the calculus becomes a rigorous science." – A.–L. Cauchy, Cours d'Analyse (1821)

"The calculus was the first achievement of modern mathematics and it is difficult to overestimate its importance. I think it defines more unequivocally than anything else the inception of modern mathematics; and the system of mathematical analysis, which is its logical development, still constitutes the greatest technical advance in exact thinking." – John von Neumann

8.2 The Real Line as a Continuum

One way to view the real line is as the set of all (possibly infinite, non-repeating) decimals. Another is to view the real line as the completion of the rationals under limits of Cauchy sequences.

Definition 8.1 (Cauchy Sequence) *A sequence (x_n) of real or rational numbers is called a* Cauchy sequence *if for every $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that for all $m, n \geq N$,*

$$|x_m - x_n| < \varepsilon.$$

In other words, the terms of the sequence eventually become arbitrarily close to one another. It turns out that every Cauchy sequence converges in $\mathbb{R}$, but not necessarily in $\mathbb{Q}$, which shows that the rationals are not complete.

8.2.1 Cantor: The Reals are uncountable

"I see it, but I don't believe it!" – Georg Cantor

In 1874, Cantor used his celebrated *diagonal argument* to prove that no list can contain all real numbers in $(0, 1)$.

Theorem 8.2 (Cantor's Diagonal Argument) *The interval $(0, 1)$ is uncountable. In particular, there is no bijection between the natural numbers $\mathbb{N}$ and the real numbers in $(0, 1)$.*

Proof: Suppose, for the sake of contradiction, that $(0, 1)$ is countable. Then every real number in $(0, 1)$ has a decimal expansion of the form

$$0.a_1a_2a_3\ldots,$$

and our assumption implies that it is possible to list *all* such numbers in a single infinite sequence. Thus suppose we have an enumeration

$$x_1, x_2, x_3, \dots$$

of all real numbers in $(0, 1)$, where each x_n is written in decimal form as

$$\begin{aligned} x_1 &= 0.a_{11}a_{12}a_{13}a_{14}\dots \\ x_2 &= 0.a_{21}a_{22}a_{23}a_{24}\dots \\ &\vdots \\ x_n &= 0.a_{n1}a_{n2}a_{n3}a_{n4}\dots \\ &\vdots \end{aligned}$$

Here $a_{nk} \in \{0, 1, 2, \dots, 9\}$ is the k-th digit of the n-th number in the list.

Important convention. To avoid the ambiguity of decimal expansions such as $0.2499999\ldots = 0.25$, we assume that no number in the list ends in an infinite tail of 9's. Every real number in $(0, 1)$ has at least one representation without trailing 9's, so this assumption does not omit any real number.

Now we construct a new number y whose decimal digits differ from the diagonal digits of the list. Define $y = 0.b_1b_2b_3\dots$ by

$$b_k = \begin{cases} 5, & \text{if } a_{kk} \neq 5, \\ 4, & \text{if } a_{kk} = 5. \end{cases}$$

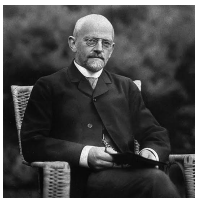
In other words, b_k is chosen so that it *differs* from a_{kk}. This is always possible because there are ten possible digits, and we deliberately avoid choosing the digit 9, so y will not end in an infinite string of 9's.

Consider any index $n \in \mathbb{N}$. We compare y with the n-th number x_n in the list. The n-th digit of x_n is a_{nn}, while the n-th digit of y is b_n, which was chosen to be *different* from a_{nn}. Therefore,

$$y \neq x_n.$$

Since this argument holds for every n, we conclude that the constructed number y is different from x_n for all $n \in \mathbb{N}$. But this contradicts our assumption that $(0, 1)$ was completely enumerated by the sequence $x_1, x_2, x_3, \dots$.

Thus our original assumption was false. No such complete enumeration exists, and therefore $(0, 1)$ is uncountable. □



"No one shall expel us from the paradise that Cantor has created." – David Hilbert

The result was revolutionary and laid the foundations of modern set theory. Quite amazingly, *diagonalization* as a proof technique has also turned out to be foundational in logic and computer science.

8.3 Algebraic and Transcendental Numbers

The irrationals split into two classes.

A real number x is *algebraic* if it is a root of (a solution of) an equation like

$$a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0 = 0$$

with rational coefficients $a_0, a_1, \ldots, a_n$. Expressions such as the left hand side of the above equation are called *polynomials* and the solutions are called its *roots*.

Some examples are

$$\sqrt{2}, \qquad \sqrt[3]{5}, \qquad \frac{1+\sqrt{5}}{2}$$

If a number is not algebraic, it is *transcendental*. Some famous examples are e and π which will soon discuss.

Liouville (1844) constructed numbers that are "too well approximated" by rationals:

$$L = \sum_{k=1}^{\infty} 10^{-k!}$$

Liouville proved that L is transcendental.

> "Transcendental numbers are not rare; they are, in fact, the rule." — *Kurt Mahler*

8.3.1 How fast do our money grow? Simple and compound interests

When we invest in a financial instrument our money earns *interest* and grows.

Simple interest assumes that interest is calculated only on the original principal, and that previously earned interest does not itself earn further interest.

Let

$$P = \text{principal (initial amount)}, \qquad r = \text{annual interest rate}, \qquad t = \text{time in years}.$$

Under simple interest, the total amount after t years is

$$A(t) = P(1 + rt).$$

This is a linear function of t. The amount increases at a constant rate Pr per year, so the graph of $A(t)$ against t is a straight line.

Example 8.1 *Suppose you invest*

$$P = ₹10{,}000$$

at an annual simple interest rate of $r = 0.08$ *(i.e. 8% per year). Then*

After 1 year:	$A(1) = 10000(1 + 0.08) = 10800$
After 3 years:	$A(3) = 10000(1 + 0.08 \times 3) = 10000(1 + 0.24) = 12400$
After 10 years:	$A(10) = 10000(1 + 0.08 \times 10) = 10000(1 + 0.80) = 18000$

In each year, the amount increases by the same amount:

$$Pr = 10000 \times 0.08 = ₹800$$

Under *compound interest*, interest is added periodically and in each period the previously earned interest itself earns further interest. If P is the principal, r is the annual interest rate, and interest is compounded once per year, then the amount after t years is

$$A_c(t) = P(1 + r)^t.$$

Example 8.2 *For the same values as in the simple–interest example,*

$$P = 10{,}000, \qquad r = 0.08,$$

we obtain

$$A_c(t) = 10000(1.08)^t$$

Thus,

After 1 year:	$A(1) = 10000(1.08) = 10800$
After 3 years:	$A(3) = 10000(1.08)^3 = 12597.12$
After 10 years:	$A(10) = 10000(1.08)^10 = 21589.25$

While simple interest grows linearly, compound interest grows exponentially. Over long durations the difference becomes significant, and compounded growth eventually "pulls away" from linear growth.

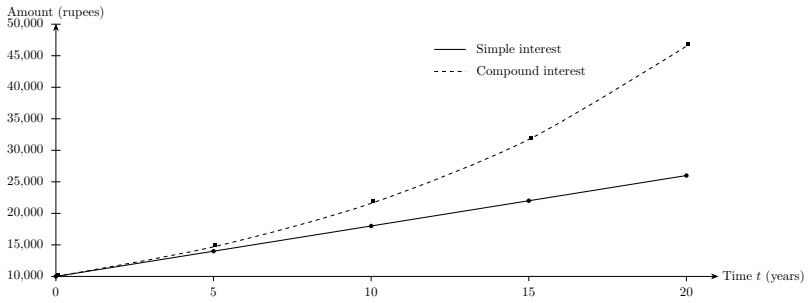


Figure 8.1: Simple vs annually compounded growth at 8% for $P = 10{,}000$ over 20 years. Vertical scale: $y = (A(t) - 10000)/5000$.

As n increases, this expression approaches the continuously compounded growth

$$A(t) = Pe^{rt},$$

revealing the natural appearance of the constant e. Thus the transition from simple to compound interest provides a natural pathway from linear to exponential growth.

8.3.2 From compound interest to the number e

Compound interest provides a natural setting in which the exponential number e appears. Suppose a principal amount P is invested at an annual interest rate r, but interest is now credited not once a year, but n times per year. In each compounding period the interest rate is r/n, so the amount after one year becomes

$$A_n(1) = P\left(1 + \frac{r}{n}\right)^n.$$

For example:

- with $n = 1$ (annual compounding), we obtain $P(1 + r)$,
- with $n = 4$ (quarterly compounding), we obtain $P\left(1 + \frac{r}{4}\right)^4$,
- with $n = 12$ (monthly compounding), we obtain $P\left(1 + \frac{r}{12}\right)^{12}$,
- with $n = 365$ (daily compounding), we obtain $P\left(1 + \frac{r}{365}\right)^{365}$.

As n increases, interest is credited more and more frequently, and the amount grows slightly larger. A natural mathematical question is:

$$\textit{What happens as } n \to \infty?$$

To study this limit, let us first focus on the core expression

$$\left(1+\frac{1}{n}\right)^n.$$

One can check numerically that

$$\left(1+\frac{1}{n}\right)^n = 2.0000,\ 2.2500,\ 2.3707,\ 2.3979,\ 2.7048,\ \ldots$$

for $n = 1, 2, 3, 4, 10, \ldots$, and that the sequence increases and approaches a limiting value. This limit is denoted by the symbol e[1].

$$e = \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n \approx 2.71828$$

The limit can also be written as an infinite sum. The full Binomial expansion becomes

$$\left(1+\frac{1}{n}\right)^n = 1+1+\frac{1}{2}\left(1-\frac{1}{n}\right)+\frac{1}{6}\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)+\cdots.$$

As $n\to\infty$, $\frac{1}{n}\to 0$, Hence,

$$e = 1+1+\frac{1}{2!}+\frac{1}{3!}+\frac{1}{4!}+\cdots$$

Thus, continuous compounding at rate r leads to the amount

$$A(1) = Pe^{\,r}$$

In general, after t years the continuously compounded amount becomes[2]

$$A(t) = Pe^{\,rt}.$$

This is the fundamental exponential-growth formula. The appearance of e is therefore not mysterious: it arises naturally as the limiting value of compound interest when interest is credited infinitely often.

> *"The number e is the unique real number for which continuous growth at rate 1 yields a unit increase in one time unit."*

This interpretation makes e one of the most important constants in mathematics, appearing in finance, probability theory, calculus, differential equations, and many dynamical processes in the natural and social sciences.

8.3.3 The Number π from Geometry

All circles are similar: any circle can be obtained from any other by a uniform scaling. Under scaling by a factor $k > 0$, every length is multiplied by k. Hence circumference and diameter scale by the same factor, so their ratio is unchanged.

Definition 8.2 *For every circle,*

$$\pi = \frac{C}{D} = \frac{C}{2r}.$$

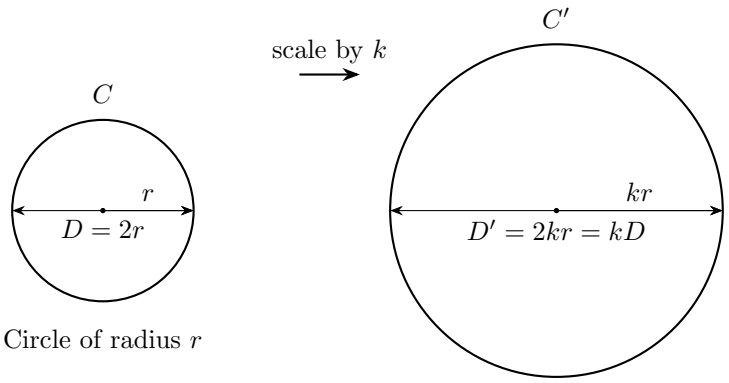
For a disk of radius r, the area is given as

$$A(r) = \pi r^2.$$

Thus $\pi = A(1)$: the area of the unit disk.

[1] Estimating its value will require some advanced techniques which an interested reader may look up.

[2] Jacob Bernoulli discovered this limit while studying compound interest in 1683. Leonhard Euler later linked it to calculus and defined it analytically.



$$C' = kC, \quad D' = kD \quad \Rightarrow \quad \frac{C'}{D'} = \frac{C}{D}.$$

Figure 8.2: Scaling a circle by factor k multiplies both circumference and diameter by k, so the ratio C/D is the same for all circles.

Approximations of π

Archimedes approximated π using inscribed and circumscribed n-gons. Using a 96-gon, he showed

$$3\frac{10}{71} < \pi < 3\frac{1}{7}.$$

The Leibniz formula for π obtained using a geometric series approximation and integration is given as

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots$$

This series converges slowly but gives an exact expression. This series is also sometimes called the Madhava-Leibniz series as it was first discovered by Madhava of Sangamagrama[3]. It can be shown that π is a transcendental number.

Problems

1. (**Incompleteness of the rationals**) Give an example of a Cauchy sequence of rational numbers that does *not* converge to any rational number. Explain why this demonstrates a "gap" in $\mathbb{Q}$.

2. (**Achilles and the Tortoise**) In the Achilles–Tortoise setup where Achilles runs at twice the speed of the tortoise, write the explicit geometric series representing Achilles' motion. Compute its sum and argue why the paradox dissolves once infinite sums are allowed.

3. (**Density**) Prove that between any two irrational numbers there exists a rational number. (Hint: scale and shift.)

4. (**Decimal expansions**) Give an example of a non-terminating repeating decimal and a non-repeating decimal. Classify each as rational or irrational and justify your answer.

5. (**Cantor's diagonal argument**) Explain why the diagonal method does not contradict the fact that decimal representations have non-uniqueness (e.g. $0.4999\ldots = 0.5$). Why does the argument still go through?

[3]See https://en.wikipedia.org/wiki/Approximations_of_π for a history of various approximations of π.

6. (**Countability vs uncountability**) Show that the set of all infinite binary sequences is uncountable by giving a diagonal proof analogous to Cantor's proof for $(0, 1)$.

7. (**Algebraic vs transcendental**) Prove that the number

$$L = \sum_{k=1}^{\infty} 10^{-k!}$$

is irrational. (Hint: compare L with rational numbers truncated after $n!$ decimal places.)

8. (**From compound interest to** e) Compute $\left(1 + \frac{1}{n}\right)^n$ for $n = 1, 2, 5, 10, 100, 1000$. Does the sequence appear to converge? How does this motivate the definition of e?

9. (e **as a limit**) Show that the sequence

$$a_n = \left(1 + \frac{1}{n}\right)^{n+1}$$

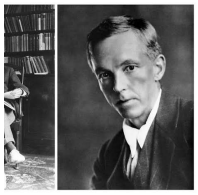
is bounded above by e. (Hint: use monotonicity and the binomial expansion.)

10. (**Archimedes' method**) Using a circumscribed n-gon around a unit circle, derive an upper bound on π. Compute the bound explicitly for $n = 6$ and $n = 12$.

11. (**Transcendentals exist "in abundance"**) Show that the set of algebraic numbers is countable. Conclude that "almost all" real numbers are transcendental.

Chapter 9

The Real Toolkit: Sequences, Functions, and Equations



"A mathematician, like a painter or a poet, is a maker of patterns" – G. H. Hardy

The real numbers are more than a continuum filling the gaps left by the rationals—they form a versatile algebraic toolkit for describing change, predicting patterns, and solving practical problems. In this chapter we develop the basic algebra of the reals as it appears in everyday mathematical modelling: from arithmetic and geometric progressions; from polynomials and their remarkable power of unique interpolation to the curve-fitting techniques that underlie modern data analysis; and from real-valued functions to systems of linear equations that allow us to solve for unknowns in structured ways. Together, these ideas illustrate how the algebra of the real numbers becomes a unifying language for patterns, growth, and relationships in both mathematics and the world around us.

9.1 Arithmetic and geometric progressions

Arithmetic and geometric progressions are two of the simplest infinite families of sequences, and they sit at the gateway between discrete sums and the idea of limits. They also provide the first nontrivial examples of closed-form formulas for sums.

9.1.1 Arithmetic progressions (AP)

Definition 9.1 *A sequence (a_n) is an* arithmetic progression *if the difference between consecutive terms is constant. That is, there exists a real number d such that*

$$a_{n+1} - a_n = d \quad \textit{for all } n \geq 1.$$

The number d is called the common difference.

Hence an AP has the form

$$a,\ a+d,\ a+2d,\ \ldots,\ a+(n-1)d,\ \ldots$$

and the n-th term is

$$a_n = a + (n-1)d.$$

Sum of the first n terms: a double-counting derivation

Let

$$S_n = a + (a+d) + (a+2d) + \cdots + (a+(n-1)d)$$

be the sum of the first n terms.

Write the same sum in reverse order:

$$S_n = (a+(n-1)d) + (a+(n-2)d) + \cdots + (a+d) + a.$$

Now add these two expressions term-by-term. Each pair adds to the same value[1]:

$$\big(a+(k-1)d\big) + \big(a+(n-k)d\big) = 2a + (n-1)d.$$

There are n such pairs, so

$$2S_n = n\big(2a + (n-1)d\big).$$

Therefore,

$$\boxed{S_n = \frac{n}{2}\big(2a + (n-1)d\big) = \frac{n}{2}\,(a_1 + a_n).}$$

Example 9.1 (Sum of the first n natural numbers) *Here $a = 1$ and $d = 1$, so*

$$S_n = \frac{n}{2}(2 + (n-1)) = \frac{n(n+1)}{2}.$$

Example 9.2 (Sum of the first n odd numbers) *The odd numbers form an AP:*

$$1, 3, 5, \ldots, (2n-1),$$

with $a = 1$, $d = 2$. Thus

$$S_n = \frac{n}{2}\big(2 \cdot 1 + (n-1) \cdot 2\big) = \frac{n}{2}(2n) = n^2.$$

So the sum of the first n odd numbers is n^2.

9.1.2 Geometric progressions (GP)

Definition 9.2 *A sequence (g_n) is a* geometric progression *if the ratio between consecutive terms is constant. That is, there exists a real number r such that*

$$\frac{g_{n+1}}{g_n} = r \quad \textit{for all } n \geq 1,$$

provided $g_n \neq 0$. The number r is called the common ratio.

A GP has the form

$$a,\ ar,\ ar^2,\ \ldots,\ ar^{n-1},\ \ldots$$

and the n-th term is

$$g_n = ar^{n-1}.$$

[1] This "pairing" trick is often attributed to the young Gauss, who is said to have summed $1 + 2 + \cdots + 100$ instantly by noticing that $1 + 100 = 2 + 99 = \cdots = 101$.

Sum of the first n terms

Let

$$S_n = a + ar + ar^2 + \cdots + ar^{n-1}.$$

Multiply both sides by r:

$$rS_n = ar + ar^2 + \cdots + ar^n.$$

Subtract:

$$S_n - rS_n = a - ar^n.$$

Hence

$$(1-r)S_n = a(1-r^n),$$

and therefore, if $r \neq 1$,

$$\boxed{S_n = a\,\frac{1-r^n}{1-r}.}$$

When $r = 1$, the GP is constant and $S_n = na$.

Infinite geometric series

If $|r| < 1$, then $r^n \to 0$ as $n \to \infty$, so

$$S_\infty = \lim_{n\to\infty} S_n = \lim_{n\to\infty} a\,\frac{1-r^n}{1-r} = \frac{a}{1-r}.$$

Thus

$$\boxed{a + ar + ar^2 + \cdots = \frac{a}{1-r} \quad (|r| < 1).}$$

This is one of the earliest places where limits naturally enter: an infinite sum makes sense only because the partial sums converge.

Example 9.3 (Zeno's geometric series)

$$\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots$$

is a GP with $a = \frac{1}{2}$ *and* $r = \frac{1}{2}$*, so*

$$S_\infty = \frac{1/2}{1 - 1/2} = 1.$$

This is exactly the mathematical resolution of Zeno's paradox.

Example 9.4 (A repeating decimal as a GP)

$$0.\overline{3} = 0.3 + 0.03 + 0.003 + \cdots$$

is a GP with $a = 0.3$ *and* $r = 0.1$*. Hence*

$$0.\overline{3} = \frac{0.3}{1 - 0.1} = \frac{0.3}{0.9} = \frac{1}{3}.$$

Remarks

- AP sums grow *quadratically* in n when $d \neq 0$, reflecting linear increments.
- GP sums grow *exponentially* in n when $|r| > 1$, reflecting multiplicative change.
- The infinite GP formula is the first rigorous example of a convergent infinite series, and is a key bridge to calculus.

9.2 Polynomials

Polynomials are among the most fundamental objects in algebra. They form a rich class of functions that are easy to compute with, easy to approximate, and powerful enough to model a wide variety of phenomena. In this section we introduce the basic theory of polynomials, discuss the unique interpolation theorem, and show how polynomials naturally arise in curve fitting and other applications.

9.2.1 Definition and degree

A *polynomial* over the real numbers is a function of the form

$$P(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n,$$

where $a_0, a_1, \ldots, a_n \in \mathbb{R}$ and n is a nonnegative integer. The number n is the *degree* of the polynomial (provided $a_n \neq 0$).

- A polynomial of degree 0 is a constant function.
- A polynomial of degree 1 is a linear function. It gives the equation of a straight line.
- A polynomial of degree 2 is a quadratic function.
- A polynomial of degree 3 is a cubic function, and so on.

The set of all polynomials with real coefficients is denoted by $\mathbb{R}[x]$. Polynomials are defined everywhere on $\mathbb{R}$. They also satisfy the useful property that if

$$P(x) = 0 \quad \text{for infinitely many distinct } x \in \mathbb{R},$$

then P must be the zero polynomial. This is called the *identity theorem for polynomials*.

9.2.2 Basic algebraic properties

Polynomials can be added and multiplied term-by-term, following the same rules as with ordinary algebraic expressions. Here we illustrate both operations with simple examples.

$$\deg(P+Q) \leq \max(\deg P, \deg Q), \qquad \deg(PQ) = \deg P + \deg Q,$$

unless cancellation occurs in the leading terms.

Polynomial addition

Let

$$P(x) = 3x^3 - 2x + 5, \qquad Q(x) = 4x^2 + x - 1.$$

To compute $P(x) + Q(x)$, we combine like terms:

$$\begin{aligned} P(x) + Q(x) &= (3x^3) + (4x^2) + (-2x + x) + (5 - 1) \\ &= 3x^3 + 4x^2 - x + 4. \end{aligned}$$

Thus

$$\boxed{P(x) + Q(x) = 3x^3 + 4x^2 - x + 4.}$$

Polynomial multiplication

Let

$$A(x) = 2x^2 - 3x + 1, \qquad B(x) = x - 4.$$

Multiplying out (using distributivity):

$$\begin{aligned} A(x)B(x) &= (2x^2)(x) + (2x^2)(-4) + (-3x)(x) + (-3x)(-4) + (1)(x) + (1)(-4) \\ &= 2x^3 - 8x^2 - 3x^2 + 12x + x - 4. \end{aligned}$$

Now combine like terms:

$$A(x)B(x) = 2x^3 - 11x^2 + 13x - 4.$$

So

$$\boxed{A(x)B(x) = 2x^3 - 11x^2 + 13x - 4.}$$

9.2.3 The decimal system as a polynomial representation

Every finite decimal expansion

$$d_k d_{k-1} \cdots d_1 d_0$$

represents a number that can be interpreted as a polynomial in 10:

$$d_k 10^k + d_{k-1} 10^{k-1} + \cdots + d_1 10 + d_0.$$

In this sense, the decimal numeral system is simply a *polynomial in the base* 10.

Example 9.5 *1. A whole number:*

$$374 = 3 \cdot 10^2 + 7 \cdot 10^1 + 4 \cdot 10^0.$$

2. A larger number:

$$5{,}182 = 5 \cdot 10^3 + 1 \cdot 10^2 + 8 \cdot 10 + 2.$$

3. A decimal (fractional) number: For digits after the decimal point, we use negative powers of 10:

$$46.327 = 4 \cdot 10^1 + 6 \cdot 10^0 + 3 \cdot 10^{-1} + 2 \cdot 10^{-2} + 7 \cdot 10^{-3}.$$

Thus every decimal numeral is nothing more than a polynomial expression in the base 10 with coefficients drawn from $\{0, 1, \ldots, 9\}$. This viewpoint is fundamental in computer science (base 2), engineering (base 16), and number theory (general base-b expansions).

Exercise 9.1 *Explain the junior school algorithm for integer multiplication in terms of polynomial multiplications. Explain how is this repeated addition?*

9.2.4 The unique interpolation theorem

One of the most remarkable facts about polynomials is that they can be used to fit any finite collection of data points with a unique polynomial of sufficiently small degree.

Theorem 9.1 (Unique Interpolating Polynomial) *Let $(x_0, y_0), (x_1, y_1), \ldots, (x_n, y_n)$ be $n+1$ data points with*

$$x_i \neq x_j \quad \textit{whenever } i \neq j.$$

Then there exists a polynomial $P(x)$ of degree at most n such that

$$P(x_i) = y_i \quad \textit{for all } i = 0, 1, \ldots, n,$$

and this polynomial is unique.

Proof: We prove existence and uniqueness.

Existence (Lagrange construction)

For each $k \in \{0, 1, \ldots, n\}$ define the *Lagrange basis polynomial*

$$L_k(x) = \prod_{\substack{0 \le j \le n \\ j \ne k}} \frac{x - x_j}{x_k - x_j}.$$

This is well-defined because $x_k - x_j \neq 0$ for $j \neq k$.

Claim 1. For all $i, k \in \{0, \ldots, n\}$,

$$L_k(x_i) = \delta_{ik} = \begin{cases} 1, & i = k, \\ 0, & i \neq k. \end{cases}$$

Proof of Claim 1. If $i = k$, then every factor in the product equals

$$\frac{x_k - x_j}{x_k - x_j} = 1,$$

so $L_k(x_k) = 1$. If $i \neq k$, then the product defining L_k contains the factor with $j = i$, namely

$$\frac{x_i - x_i}{x_k - x_i} = 0,$$

so $L_k(x_i) = 0$. This proves the claim. □

Now define

$$P(x) = \sum_{k=0}^{n} y_k \, L_k(x).$$

Each L_k is a polynomial of degree n (a product of n linear factors), so P is a polynomial of degree at most n.

Evaluate P at each interpolation point x_i:

$$P(x_i) = \sum_{k=0}^{n} y_k \, L_k(x_i) = \sum_{k=0}^{n} y_k \, \delta_{ik} = y_i.$$

Hence P interpolates all the given data. Existence is proved.

Uniqueness

We need one standard fact.

Lemma. A nonzero polynomial of degree at most n can have at most n distinct real roots.
Proof of Lemma. We proceed by induction on n. For $n = 0$, a nonzero constant polynomial has no roots. Assume the statement true for degree at most $n - 1$. Let Q be a nonzero polynomial of degree at most n with a root a. Then $(x - a)$ divides $Q(x)$, so

$$Q(x) = (x - a)R(x)$$

for some polynomial R of degree at most $n - 1$. Any root of Q different from a must be a root of R, and by the induction hypothesis R has at most $n - 1$ distinct roots. Thus Q has at most n distinct roots in total. □

Now suppose P and Q are two polynomials of degree at most n both interpolating the data:

$$P(x_i) = Q(x_i) = y_i \quad \text{for all } i = 0, \ldots, n.$$

Consider their difference

$$R(x) = P(x) - Q(x).$$

Then $\deg R \leq n$, and

$$R(x_i) = P(x_i) - Q(x_i) = 0 \quad \text{for all } i = 0, \ldots, n.$$

So R has at least $n+1$ distinct roots. By the lemma, the only polynomial of degree at most n that can have $n+1$ roots is the zero polynomial. Hence $R \equiv 0$, i.e.

$$P(x) = Q(x) \quad \text{for all } x.$$

Therefore the interpolating polynomial is unique. □

A simple example

Given

$$(0, 1), \qquad (1, 2), \qquad (2, 5),$$

there is a unique quadratic polynomial through these points. Using the Lagrange formula,

$$P(x) = 1 \cdot \frac{(x-1)(x-2)}{(0-1)(0-2)} + 2 \cdot \frac{(x-0)(x-2)}{(1-0)(1-2)} + 5 \cdot \frac{(x-0)(x-1)}{(2-0)(2-1)}.$$

Simplifying yields

$$P(x) = x^2 + 1.$$

9.2.5 Polynomials and curve fitting

Interpolation provides an exact polynomial passing through given data, but we often only have noisy measurements or more data points than we want the degree of the polynomial to match. In practice, measured data points rarely lie exactly on a straight line, a quadratic curve, or any simple mathematical model. We therefore need a way to choose a "best" curve that captures the overall trend of the data, without requiring a perfect fit to every point. The most widely used approach is the *method of least squares*, which selects the curve whose predicted values come, on average, as close as possible to the observations.

The idea is simple to state: among all candidate lines or curves, choose the one that *minimises the total squared error*

$$\sum_{i=1}^{n} (y_i - P(x_i))^2,$$

where $P(x)$ is the model we want to fit. Squaring the errors penalises large deviations more strongly and leads to a unique, stable solution. Although the derivation is not required here, the resulting formulas give clean, explicit expressions for the best-fitting straight line or quadratic polynomial.

Historically, the method was developed by Carl Friedrich Gauss in the early 19th century in his work on determining planetary orbits from noisy astronomical measurements. Gauss famously used least squares to recover the orbit of the asteroid *Ceres* in 1801, demonstrating that the best-fitting curve—in the least-squares sense—provides the most reliable prediction from imperfect data. Today the same principle forms the basis of regression, statistical modelling, and modern machine learning.

We will not describe his methods in these notes and leave it for a future course. In what follows we give some examples to illustrate the use of least squares approximation.

Example 9.6 (Linear Fit Example (Economics): Wage vs. Experience) *A common empirical observation in labour economics is that, over an early career window, wages rise approximately linearly with experience. Suppose we measure monthly wage W (in thousand rupees) for workers with different years of experience x:*

Experience x (years)	*0*	*2*	*4*	*6*	*8*	*10*
Wage W (₹'000/month)	*30*	*38*	*46*	*55*	*63*	*70*

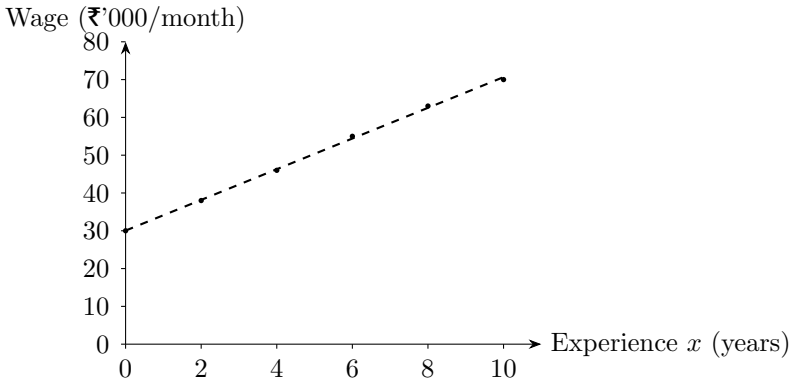


Figure 9.1: Wage vs. experience.

We model this by a straight line

$$W(x) \approx mx + c.$$

A least–squares calculation gives (See Figure 9.1)

$$m \approx 4.06, \qquad c \approx 30.05,$$

so the fitted wage equation is

$$\boxed{W(x) \approx 4.06x + 30.05.}$$

Interpretation. *The slope means that each additional year of experience raises expected wage by about ₹4,060 per month in this range. The intercept estimates a starting wage around ₹30,000 per month.*

Example 9.7 (Quadratic Fit Example (Economics): Profit vs. Output) *In many production settings, profits rise with output at first, but beyond some point diminishing returns and rising costs reduce profits. This often produces a concave (downward-opening) quadratic relationship.*

Suppose a firm varies daily output q (in hundreds of units) and observes profit Π (in thousand rupees):

Output q (hundreds)	*0*	*1*	*2*	*3*	*4*	*5*	*6*
Profit Π (₹'000)	*0*	*8*	*14*	*18*	*20*	*19*	*15*

We fit a quadratic model

$$\Pi(q) \approx aq^2 + bq + c, \qquad a < 0.$$

Least squares gives (See Figure 9.2)

$$a \approx -1.18, \qquad b \approx 9.68, \qquad c \approx -0.29,$$

so

$$\boxed{\Pi(q) \approx -1.18q^2 + 9.68q - 0.29.}$$

Interpretation. *The negative coefficient a reflects diminishing returns. The profit-maximising output is near the vertex:*

$$q^* = -\frac{b}{2a} \approx \frac{9.68}{2.36} \approx 4.1,$$

i.e., roughly 4.1 hundred units per day.

Polynomials offer some of the simplest yet most expressive "laws of variation" in mathematics, and they occur in many different contexts.

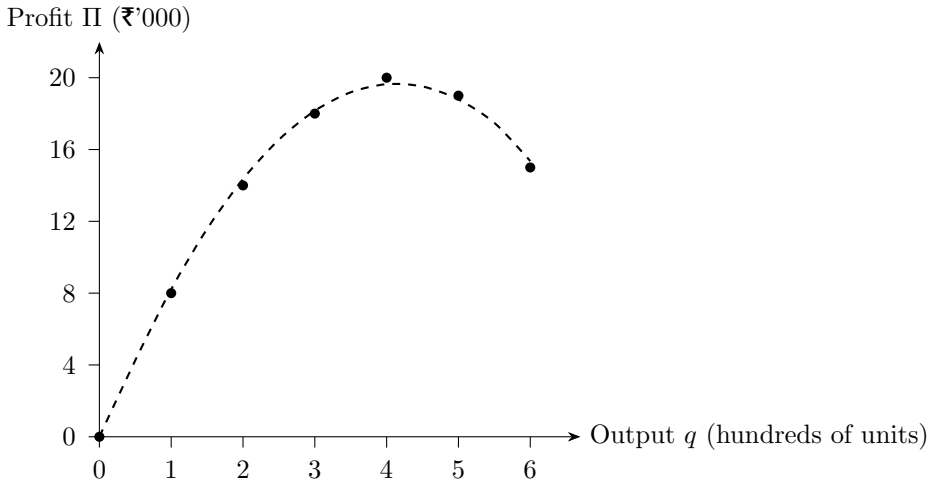


Figure 9.2: Profit vs. output showing a concave quadratic least-squares fit.

9.3 Systems of linear equations

A system of linear equations is a collection of m equations in n unknowns $x_1, x_2, \ldots, x_n$ of the form

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{in}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\ \vdots \qquad\qquad\qquad\qquad\quad &\;\;\vdots \;\; \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= b_m \end{aligned}$$

where the a_{ij} and b_i are known real numbers and the x_j are the unknowns. In general, if the equations are independent (none can be obtained as a combination of the others) and $m = n$ (as many equations as unknowns), then there is a unique solution for the x_is.

Linear systems are among the oldest and most fundamental objects in algebra: they form the basis of modern numerical computation, optimization, statistics, and scientific modelling. Despite their simple appearance, linear systems encode a remarkable variety of real-world relationships.

9.3.1 Why linear equations matter

Even when the underlying model is complicated or nonlinear, it is often approximated by a linear system. This process, called *linearisation*, relies on the fact that over sufficiently small ranges many functions behave almost linearly. Because linear equations can be solved efficiently, stably, and exactly by well-understood methods, they serve as the computational workhorse of applied mathematics.

Linear systems also arise naturally whenever several quantities influence one another additively. In economics, physics, and engineering, additive effects are frequently easier to measure or estimate, so linear models become the default starting point for analysis. In higher dimensions, linear systems form the algebraic language of vectors, matrices, and multivariable functions which an interested reader may study in a later course.

Example 9.8 (A simple example puzzle) *In a lab there are two kinds of metal blocks, type A and type B. Let x be the weight (in kg) of one A-block and y the weight of one B-block.*

- *Three A-blocks and two B-blocks weigh* 18 *kg.*
- *Two A-blocks and five B-blocks weigh* 24 *kg.*

Question: *Find x and y.*

Solution. *The statements translate directly into the linear system*

$$3x + 2y = 18,$$
$$2x + 5y = 24.$$

Multiply the first equation by 5 *and the second by* 2*:*

$$15x + 10y = 90,$$
$$4x + 10y = 48.$$

Subtracting gives

$$11x = 42 \quad \Rightarrow \quad x = \frac{42}{11}.$$

Substitute into $3x + 2y = 18$*:*

$$3 \cdot \frac{42}{11} + 2y = 18 \quad \Rightarrow \quad \frac{126}{11} + 2y = \frac{198}{11} \quad \Rightarrow \quad 2y = \frac{72}{11} \quad \Rightarrow \quad y = \frac{36}{11}.$$

$$\boxed{x = \frac{42}{11} \approx 3.82 \ kg, \qquad y = \frac{36}{11} \approx 3.27 \ kg.}$$

This illustrates a key point: linear systems arise whenever several unknown quantities contribute additively to measured outcomes. The same algebraic tool solves problems in mixtures, balances, circuits, flows, and data fitting.

9.3.2 Some real-world examples of linear systems

1. Economics: input–output models. In Leontief's input–output theory, each industry requires fixed proportions of output from other industries. If x_j is the total output of industry j, linear balance equations

$$x_i = a_{i1}x_1 + a_{i2}x_2 + \cdots + a_{in}x_n + d_i$$

relate production to inter-industry consumption and final demand d_i. This leads directly to a system of linear equations solved for $x_1, \ldots, x_n$.

2. Electronics: nodal analysis in circuits. Using Kirchhoff's current laws, the voltages $V_1, \ldots, V_n$ at electrical nodes satisfy

$$\sum_j G_{ij}(V_i - V_j) = I_i,$$

where G_{ij} are conductances and I_i are source currents. The result is a linear system $GV = I$, solved billions of times per second inside circuit simulators.

3. Data analysis: linear regression. A straight-line fit $y = mx + c$ to data (x_i, y_i) leads to the normal equations

$$\begin{pmatrix} \sum x_i^2 & \sum x_i \\ \sum x_i & n \end{pmatrix} \begin{pmatrix} m \\ c \end{pmatrix} = \begin{pmatrix} \sum x_i y_i \\ \sum y_i \end{pmatrix},$$

a 2×2 linear system whose solution yields the least-squares fit.

4. Physics and engineering: equilibrium conditions. Static equilibrium of forces in structures—bridges, frames, trusses—is governed by equations of the form

$$\sum F_x = 0, \qquad \sum F_y = 0,$$

which, when decomposed by components, give linear relations between unknown tensions and forces.

Summary

Systems of linear equations are central because:

- they model many real-world relationships directly,
- they provide the local approximation to nonlinear systems,
- they admit efficient and reliable solution methods,
- and they unify ideas across economics, science, engineering, and data analysis.

They form the algebraic backbone of the "real toolkit" of functions, approximations, and computational methods.

9.4 Some fascinating real-valued functions

Real-valued functions are the language in which we describe patterns, growth, decay, saturation, symmetry, oscillation, and randomness. In this section we collect a few of the most important and beautiful functions that appear repeatedly across mathematics, the sciences, economics, and data analysis. For each function we list salient properties and a couple of canonical uses.

9.4.1 The quadratic function

$$f(x) = ax^2 + bx + c, \qquad a \neq 0.$$

Salient properties.

- Graph is a parabola opening upward if $a > 0$ and downward if $a < 0$.
- Vertex at
$$x^* = -\frac{b}{2a}, \qquad f(x^*) = c - \frac{b^2}{4a}.$$
- Symmetric about $x = x^*$.

Uses.

- Uniform acceleration: $s(t) = \frac{1}{2}gt^2 + vt + s_0$.
- Concave approximations in economics (diminishing returns).

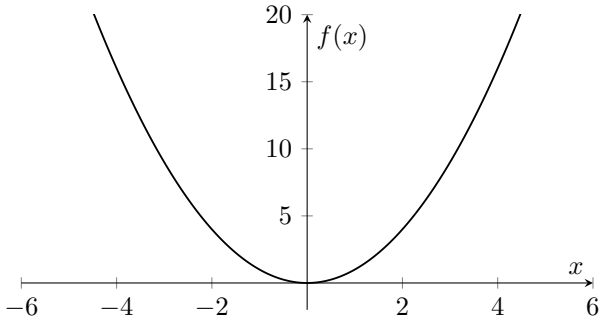


Figure 9.3: A quadratic: $f(x) = x^2$.

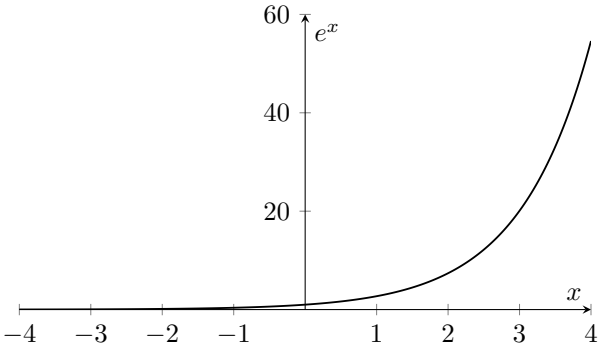


Figure 9.4: Exponential growth: $f(x) = e^x$.

9.4.2 The exponential function

$$f(x) = e^x.$$

Properties: always positive, strictly increasing, derivative equals itself.
Uses: population growth, compound interest, differential equations.

9.4.3 The negative exponential

$$f(x) = e^{-x}.$$

Properties: positive, strictly decreasing, approaches 0 as $x \to \infty$.
Uses: radioactive decay, discounting, cooling.

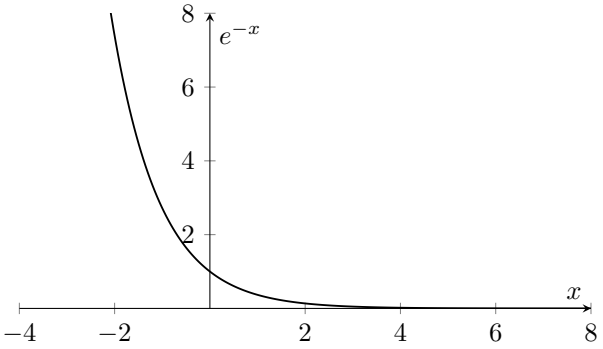


Figure 9.5: Exponential decay.

9.4.4 The logarithm

$$f(x) = \ln x, \qquad x > 0.$$

Properties: strictly increasing, slow growth, inverse of e^x.
Uses: multiplicative-to-additive conversion, data compression, statistics.

9.4.5 The sinusoid

$$f(x) = \sin x.$$

Properties: periodic, oscillatory, bounded, smooth.
Uses: waves, AC electricity, tides, Fourier series.

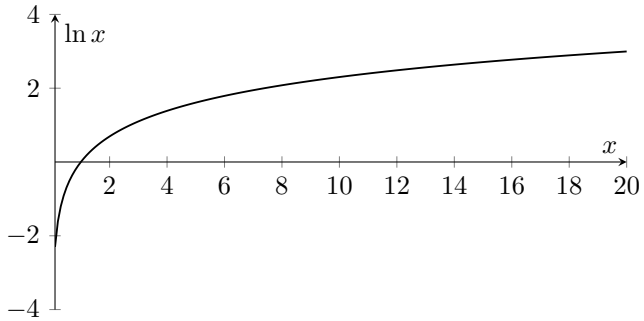


Figure 9.6: Logarithmic growth: $f(x) = \ln x$.

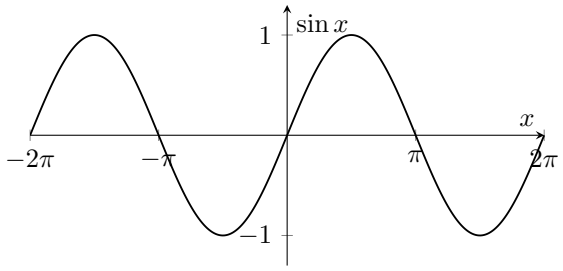


Figure 9.7: A sinusoidal oscillation: $f(x) = \sin x$.

9.4.6 The Gaussian (normal curve)

$$f(x) = \frac{1}{\sqrt{2\pi}}\, e^{-x^2/2}.$$

Properties: symmetric, bell-shaped, rapidly decaying tails.
Uses: measurement error, statistics, noise modelling.

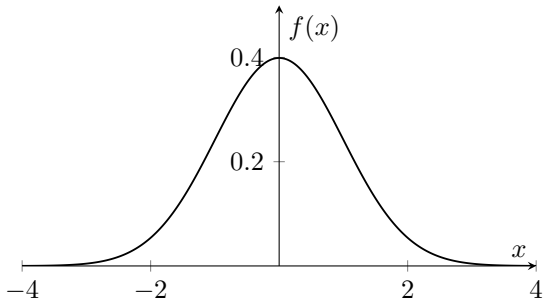


Figure 9.8: Standard Gaussian density.

9.4.7 The logistic function

$$f(x) = \frac{1}{1 + e^{-x}}.$$

Properties: S-shaped, bounded, symmetric about $x = 0$.
Uses: constrained growth, technology adoption, logistic regression.

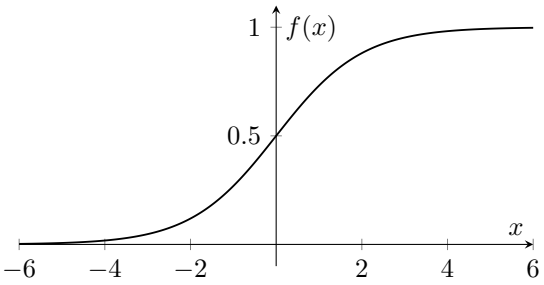


Figure 9.9: Logistic (saturation) curve.

A closing remark

These functions recur because each captures a deep "shape" that nature and society repeatedly exhibit: quadratic curvature, exponential growth and decay, logarithmic compression, sinusoidal oscillation, Gaussian randomness, and logistic saturation. Learning their properties is like acquiring a small but powerful vocabulary for the real world.

Summary

The real numbers complete the rationals, resolve ancient paradoxes, and support the modern theory of limits, continuity, and calculus. They underpin geometry, physics, finance, and modelling, and form the domain for the most important functions in science. Understanding real numbers is therefore a gateway into nearly every field of mathematics and its applications.

Problems

1. (**Arithmetic sequences**) An arithmetic progression has first term a and common difference d. Show that the average of the first n terms equals the average of the first and nth terms. Use this fact to re-derive the sum formula $S_n = \frac{n}{2}(2a + (n-1)d)$.

2. (**Geometric growth**) A bacteria culture triples every hour. If it starts with B_0 bacteria at time $t = 0$, express B_t as a geometric progression. How long does it take to exceed $50\,B_0$?

3. (**Polynomial addition and multiplication**) Let

$$p(x) = 3x^3 - 2x + 7, \qquad q(x) = x^2 - 4.$$

   Compute $p(x) + q(x)$ and $p(x) \cdot q(x)$ and state their degrees. Explain why degrees add under multiplication.

4. (**Decimal as a polynomial**) Interpret the decimal number 40756 as a polynomial in 10 evaluated at 10. Rewrite it using powers of 2 instead of 10 by expanding it in base 2.

5. (**Interpolation**) Given three points $(0, 1)$, $(1, 3)$, $(2, 6)$:

   (a) Construct the Lagrange basis polynomials $L_0(x)$, $L_1(x)$, $L_2(x)$.

   (b) Write down the interpolating polynomial explicitly.

   (c) Evaluate it at $x = 1/2$.

6. (**Overfitting with polynomials**) Explain why there always exists a polynomial of degree $n - 1$ passing through n given data points, but why such a polynomial might be a poor model for prediction.

7. (**Linear regression intuition**) Consider the data points $(0,3)$, $(1,2)$, $(2,2)$, $(3,5)$. Sketch a line that "best fits" the data by eye. Explain qualitatively why the least-squares line minimises the sum of squared vertical errors.

8. (**Systems of equations**) Solve the system:

$$\begin{cases} 3x + y = 7, \\ 2x - y = 1. \end{cases}$$

   Then interpret the solution as the intersection of two lines.

9. (**Linearity in modelling**) A factory produces two products A and B. Each unit of A requires 2 hours of machine time and 3 units of raw material. Each unit of B requires 1 hour of machine time and 4 units of raw material. If the factory uses 23 hours of machine time and 41 units of raw material in total, set up the linear system and solve for the production quantities.

10. (**Uniqueness of solutions**) Give an example of a 2×2 linear system with:

    - exactly one solution,
    - infinitely many solutions,
    - no solution.

    Explain geometrically why these cases correspond to intersecting lines, coincident lines, and parallel lines.

11. (**Function families**) For each of the functions below—quadratic, exponential, logarithmic, sinusoidal, logistic—provide:

    (a) a real-world phenomenon it models,

    (b) whether it is increasing, decreasing, periodic, or saturating,

    (c) the effect of scaling the input (e.g. replacing x by kx).

12. (**Gaussian decay**) Show that the Gaussian function

$$g(x) = e^{-x^2}$$

    decays faster than any exponential e^{-cx} as $x \to \infty$. (Hint: compare exponents.)